Production Networks and R&D Misallocation

Yasutaka Koike-Mori   Koki Okumura

UCLA   UCLA*

November 15, 2024

Preliminary

Abstract

This paper investigates the interaction between production networks and firms' research and development (R&D) decisions and their implications for aggregate inefficiency. Using Japanese inter-firm transaction and patent data, we document that older firms have more network connections and tend to connect with other older firms. Additionally, connected firms' R&D stimulates innovation in their partner firms. Motivated by these empirical findings, we construct a model that incorporates the dynamics of production networks and R&D as a new variety creation. In this model, firms gradually build their supply chains and can leverage their existing supply chains to develop and sell new products. Our model implies that older firms at the center of the production network underinvest in R&D relative to the optimal allocation.

*We thank Ariel Burstein, David Baqaee, and Hugo Hopenhayn for excellent guidance. We also thank Andy Atkeson, Oleg Itskhoki, Lee Ohanian, Jon Vogel, Pablo Fajgelbaum, Joao Guerreiro, Gary Hansen, Pierre-Olivier Weill and seminar participants at UCLA for valuable comments. This research has been conducted as a part of the TDB-CAREE project. We appreciate Teikoku Databank for granting us access to their database.

1 Introduction

Firms' research and development (R&D) activities fundamentally depend on their position in production networks and the innovation activities of their trading partners. For instance, automotive giants like GM and Toyota develop new models by integrating components from established engine and chassis suppliers with their new technologies, rather than building every component from scratch.

While extensive research has examined how production networks affect the allocation of productive resources, particularly production workers, and thereby aggregate output, the impact of production networks on the allocation of R&D resources has received less attention. This is despite the fact that firms' R&D decisions are based on their expected profits, which are determined by their position in the network and the innovation activities of their trading partners. These two important strands of literature have largely developed separately due to the difficulty in accessing relevant data on both production networks and R&D activities, as well as the complexity of modeling the dynamic interactions between them.

In this paper, we investigate how the structure of production networks affects the allocation of R&D resources across firms. Our analysis proceeds in three steps. First, we construct a unique dataset that combines firm-to-firm production network data with patent data in Japan. Second, we document empirical patterns about the dynamic relationship between production networks and R&D activities. Third, we develop a dynamic model that incorporates these empirical patterns, allowing us to study the impact of firm-level R&D decisions within networks on the macroeconomy. Our analysis reveals that the production network structure observed in the data implies a misallocation of R&D resources, with firms in the center of the network underinvesting in R&D relative to the social optimum.

To guide our model development, we begin by documenting empirical patterns using the Teikoku Databank (TDB) database, which contains information on transaction partners. These patterns can be primarily summarized by firm age. Importantly, the number of suppliers and buyers increases with firm age, but these growth rates decrease with age. Additionally, the older a firm is, the older its trading partners tend to be. These age-dependent patterns in the production network serve as important empirical regularities that ensure the tractability of our dynamic model while replicating important features of the data. Furthermore, utilizing a database of firm-level transaction networks and patent citation relationships, we find that an exogenous increase in the R&D of a trading partner firm, as measured by patents, promotes R&D in that firm.

Building on these empirical findings, we develop a dynamic model incorporating the interaction between production networks and R&D. Specifically, we introduce the age-dependent inter-firm matching process observed in the data, where successful R&D adds new products to the firm. The framework introduces an exogenous matching process that governs the deterministic accumulation of network connections between firms over time. Within this age-dependent network structure, firms engage in R&D activities. When R&D succeeds, firms can produce new products using their current suppliers and sell these products through their established customer base. This setting allows firms to endogenize both their network position and the innovative activities of their trading partners in their R&D decisions.

The pattern of inter-firm production network linkages replicated by the model has direct implications for the allocative efficiency of R&D. By comparing the decentralized equilibrium to the planner's solution, we find that older firms underinvest in R&D relative to the optimal allocation. In a network economy, the size of a firm's sales is determined not only by final demand but also by demand for intermediate goods from other firms. The fact that older firms are more interconnected and their counterparts are also older firms suggests that firms' revenue is too small due to the high degree of double marginalization. Furthermore, sellers do not internalize a positive externality to the buyer. This externality is disproportionately larger for older firms with more connections, which also encourages underinvestment by older firms. These two effects lead to underinvestment of R&D workers in firms located in the center of the production network in a decentralized equilibrium.

Related literature:

This paper connects the literature on the macroeconomic importance of networks, as typified by Acemoglu and Carvalho (2012), to the literature on R&D as an creation of new goods, for example, Romer (1990); Klette and Kortum (2004). Recent theoretical work on production networks can be divided into two categories: fixed and endogenous networks. The first category includes (Liu (2019); Baqaee and Farhi (2020); Bigio and La'O (2020); Baqaee et al. (2023); Osotimehin and Popov (2023)) and mainly considers inefficiencies in the allocation of factors of production in equilibrium with inter-firm linkages. The second category focuses on the mechanisms through which links are formed between firms (Oberfield (2018); Arkolakis et al. (2023); Eaton et al. (2022); Huneeus (2018); Bernard et al.

(2022); Acemoglu and Azar (2020). This paper is related to both and discusses how inter-firm network structure determined by age and endogenous innovation endogenously induces a misallocation of R&D resources [1].

Second, with respect to R&D misallocation, we study the role of R&D in the creation of new goods in a network economy. This is a novel mechanism that differs from the inefficiencies associated with endogenous growth of R&D studied in the literature. For example, Acemoglu et al. (2018) study different sources of inefficiency, with less innovative firms using R&D resources for fixed costs and crowding out more innovative firms. Another source of R&D misallocation is the impact on growth rates arising from differences in private and public returns on innovation spillovers. For example, Aghion et al. (2023) and Ayerst (2023) deviate from the standard growth model in which private and public returns to R&D are identical and allow for a separation between the two. Another approach, such as Liu and Ma (2023), measures misallocation using sufficient statistics to yield optimal public returns. In contrast, we study a different mechanism from these literatures. In our model, the heterogeneity of a new good arises not from the productivity that draws, but from the location of the production network in which the good is produced. Different production network structures give firms different R&D incentives, which leads to an inefficient allocation of R&D workers. This novel perspective contributes to the understanding of R&D misallocation in the context of production networks[2].

The rest of the paper is organized as follows. Section 2 presents the data and empirical findings on the relationship between production networks, firm age, and R&D. Section 3 develops a dynamic model of production networks and R&D based on these empirical findings. Section 4 analyzes the misallocation of R&D workers in the decentralized equilibrium compared to the social planner's solution. Section 5 presents a quantitative exercise, estimating the model parameters and comparing the model's predictions with the data. Finally, Section 6 concludes the paper.

[1]Research on production network are driven by empirical analyses (Boehm et al. (2019); Bernard et al. (2022, 2019); Carvalho et al. (2020); Bai et al. (2023); Daisuke (2017)) using recently available inter-firm network data. Much of the paper focuses on the spillover effects of short-term shocks through the supply chain, represented by so-called supply chain destructions.The data we construct includes information on firm-level linkages and patent data, providing the first evidence on the long-term life-cycle structure of supply chains and innovation.

[2]Aekka and Gaurav (2023), whose work is contemporaneous with ours, use the similar method as ours to solve a dynamic endogenous network model. That is, the equilibrium in their model, like ours, is also described by firm age and productivity. The difference with our study is that they consider endogenous search to connect with customers and suppliers, while we focus on firms' R&D decisions.

2 Motivating Facts on Production Network and Innovation

Using a dataset of B2B transactions and patents as an proxy of R&D, we documents the facts to govern the dynamic pattern of supply chain. Older firms more intensively use intermediate goods and have wider networks with more buyers and suppliers. These firms are also more interconnected with other older firms. Furthermore, we find that an exogenous increase in the R&D of a trading partner firm, as measured by patents, promotes R&D in that firm. Based on these empirical regularities, in section 3 we build a dynamic model.

2.1 Data Sources

Our analysis relies on three main datasets: the Teikoku Databank (TDB) dataset from Japan, the IIP Patent Database, and the OECD Triadic Patent Families (TPF) database.

First, the Teikoku Databank is a private credit research company obtains information during the process of obtaining credit research reports about potential suppliers and buyers. This information includes a series of corporate-level characteristics along with the identities of the companies' suppliers and buyers. Notably, this database is not limited to publicly listed companies, offering broader coverage compared to databases like Compustat.

The second dataset is the IIP Patent Database, developed for patent statistical analysis using standardized data from the Patent Office. As the IIP Patent Database lacks corporate information, it is linked with a NISTEP database that connects patent data to company names and identification numbers. These identification numbers from the NISTEP database are then used to merge the IIP Patent Database with the TDB database. For firms without identification numbers, the databases are integrated as much as possible using addresses and names.

The last database is the OECD Triadic Patent Families (TPF) database, which is a dataset that combines patent applications filed with the European Patent Office (EPO), the Japan Patent Office (JPO), and the United States Patent and Trademark Office (USPTO) into patent families based on common priority applications.TPF database The primary data source for the TPF is the EPO's Worldwide Patent Statistics Database, which provides harmonized and comparable information on patents from the EPO, JPO, and USPTO. We use information on the IPC (four digits), the year of application (Earliest filing date to Japan patent office) and the nationality of the applicant for each patent.

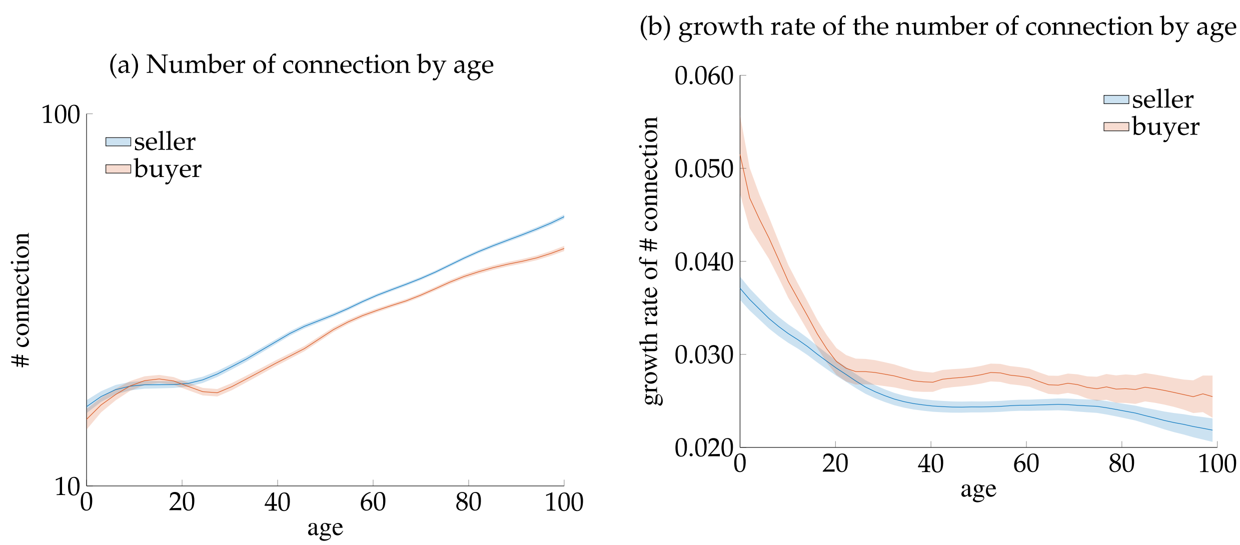
2.2 Relationship between supply chain and firm age

We explore the empirical regularities governing production networks' time evolution and cross-sectional distribution to model the inter-firm matching process. The main finding is that the number of suppliers and buyers increases with firm age, but the rate of increase decreases with age. In the cross-section, older firms are more likely to have older trading partners as well, mainly due to the stickiness of the trading relationship.

2.2.1 Age-dependent network relationships

We begin by analyzing the cross-sectional relationship between firm age and the number of trading partners. The figure 1a shows a local linear regression plot of the number of suppliers and the number of buyers versus age for a given firm. Both are shown to be monotonically increasing functions of age. After a strong increase in the first half of the life cycle, once a certain level is reached, the trend shifts to an increasing trend with a roughly log-linear relationship. Figure 1b is a local linear regression plot against the rate of change, which confirms that the rate of change is high in the first half of the life cycle and converges to a constant value, as expected. The number of suppliers tends to slightly exceed the number of buyers, but the slopes are not markedly different.

Figure 1: Lifecycle of production network.



Notes: The figure shows the local linear regression of firm age (vertical axis) on the firms connections (horizontal axis) with controls. The sample is 1998-2019. The control terms include prefecture, industry, and year fixed effects. The two lines represent the supplier and buyer side as separate regressions. Shaded area indicates the 95 percent condence intervals.

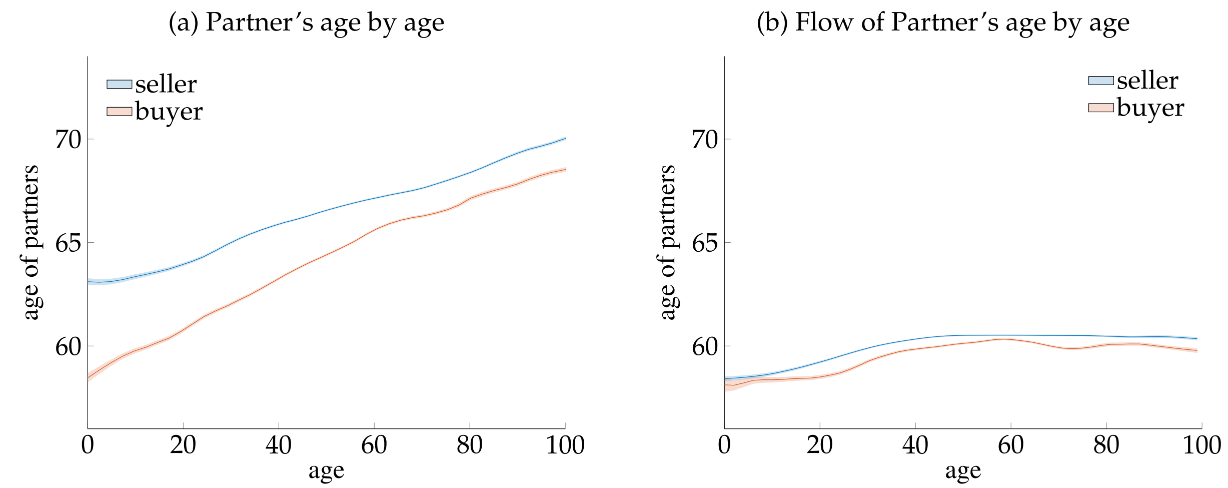
2.2.2 Positive age assortative matching by age in cross-section

Next, we turn our attention to the features of trading partners conditional on age. Figure 2a plots the age of trading partners. The higher the age of the trading partner, the higher the age of the trading partner tends to be. This indicates that positive assortative matching by age is observed in the cross-section.

To understand this assortative matching, we consider existing links (stocks) and the formation or withdrawal of business relationships (flows). Initially, we examine the age heterogeneity in current period matches among the flows. Figure 2b shows a local linear regression plot of the partner's age and one's own age linked by age. The heterogeneity of the flows themselves is only one to two years old, which is considerably smaller than the heterogeneity of the cross-section. Next, to see the age dependence of link termination, Figure 3 plots the survival of each link as a function of the number of trading years between firms for each supplier and buyer age quartile group. The probability of a link breaking is a decreasing function of the number of transaction years, but there is little significant heterogeneity with respect to the age group conditioning for each seller and buyer.

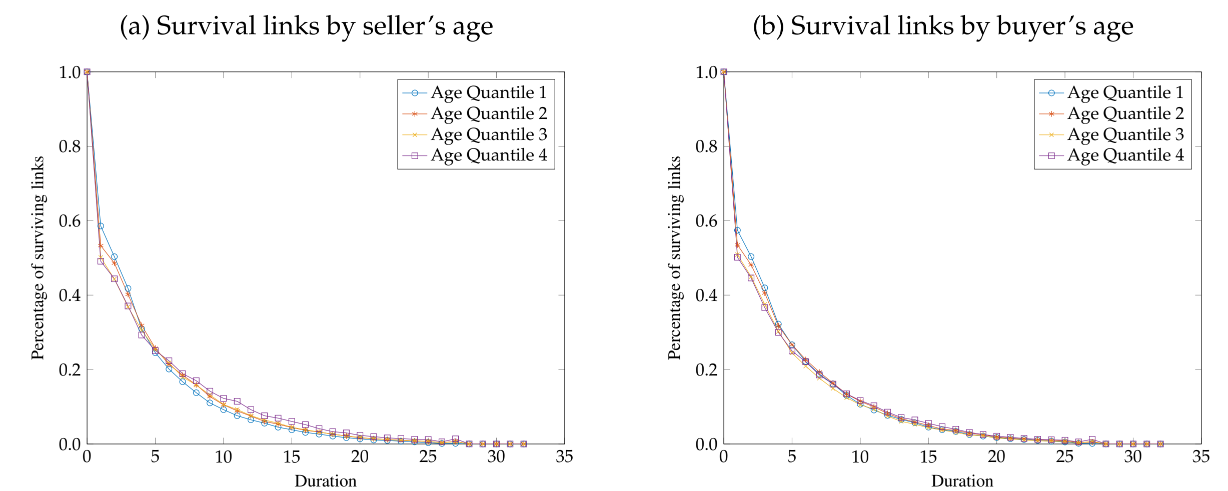
Then, why does age assortativity arise, even though the creation end of links are independent on firm age? The reason is that when business relationships are sticky, firms and their past counterparts age together. According to Figure 3, it takes more than 10 years for a business relationship to be lost until 90% of the relationship is lost. Such large stickiness of business relationships generates the observed positive age assortativity of matching.

Figure 2: Positive Age assortativity.



Notes: The figure shows the local linear regression of firm age (vertical axis) on the firms connections (horizontal axis) with controls. The sample is 1998-2019. The control terms include prefecture, industry, and year fixed effects. The two lines represent the supplier and buyer side as separate regressions. Shaded area indicates the 95 percent confidence intervals.

Figure 3: Survival links.

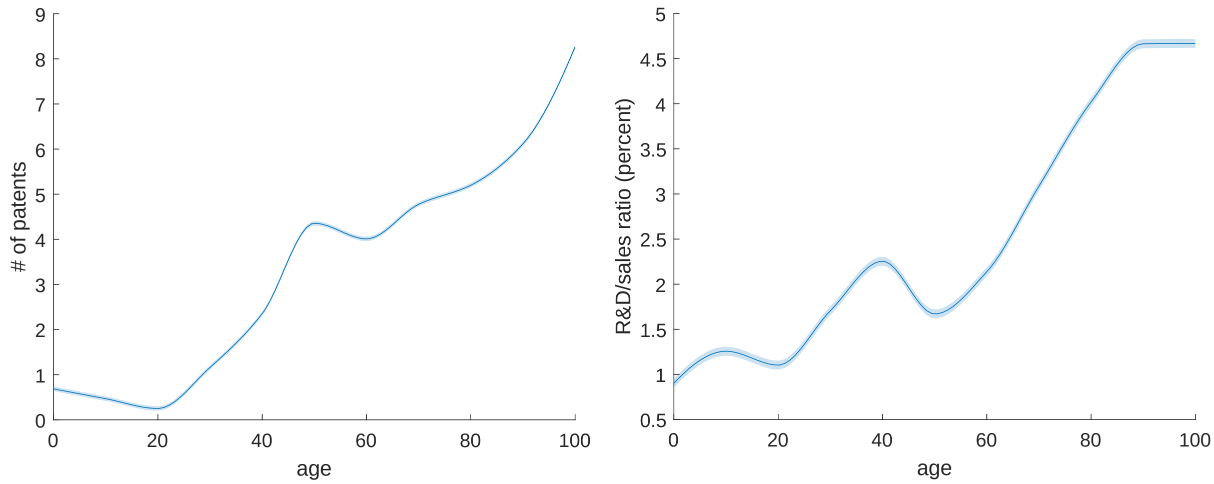


Notes: This figure plots the duration of business relationship between firms. To remove the bias caused by the observability of firms, The sample is restricted to connections where (1) the connection is first observed more than one year after period in which both firms were first observed and (2) the connection is no longer observed more than one yeare before the period in which either firm was last observed. The figure on the left classifies business relationship by the age quartiles of supplier firms. The right figure classifies business relationship by the age quartiles of the buyer firms.

Finally, the results of the local linear regression are also shown in the Figure 4. The two proxy measures of R&D activity are monotonically increasing functions of age.

Figure 4: R&D and age.

(a) The number of patents (b) R&D/Sales ratio



Notes: The figure shows the local linear regression of firm age (vertical axis) on the measures of RD activities (horizontal axis) with controls. The sample is 1998-2019. The control terms include prefecture, industry, and year fixed effects for the firm and its clients. Shaded area indicates the 95 percent confidence intervals.

2.3 Firm level Production Networks and R&D

In this section, we use an integrated database that combines firm-level transaction data from TDB and patent information from the IIP Patent Database, organized by the JPO, and investigate the impact of the R&D activities of firms connected through the production network on their own R&D activities.

While our formulation is motivated by Acemoglu et al. (2015) and Liu and Ma (2023), it differs in two key aspects. First, while they use aggregated variables by industry level or technology class level, we use firm-level transaction linkages and technological linkages measured by citation relationships. Second, Liu and Ma (2023) provide empirical evidence of network spillovers through technological links (innovation networks in their language), while Acemoglu et al. (2015) analyze foreign patent shock spillovers through production networks. Sector-level analysis may not allow for a clear interpretation of the mechanism because the linkages between production and patent networks will likely overlap. In contrast, our dataset allows us to observe firm-level transactional relationships and patent citation relationships separately, allowing us to separate the roles of the two. We report in the Appendix A.1 that the two overlap even at the firm level as well.

We conduct regression analysis to analyze the impact of the R&D activities of firms connected through the production network on their own R&D activities. This analysis utilizes firm-level data spanning from 1994 to 2019. We measure firms' R&D activities by the number of patent applications and identify the presence of connections between firms within the production network using the TDB database. Then, we estimate the following regression:

$$\log |\text{Patents}|_{i,t} = \beta_1 \log \frac{|\text{Sellers' Patents}|_{i,t}}{|\text{Sellers}|_{i,t}} + \beta_2 \log \frac{\left|\text{Buyers' Patents}\right|_{i,t}}{\left|\text{Buyers}\right|_{i,t}} + \text{controls}_{i,t} + \varepsilon_{i,t}$$

where controls include and firm and year fixed effects. Here, $|\text{Sellers' Patents}|_{i,t}\left(\left|\text{Buyers' Patents}\right|_{i,t}\right)$ represents the total number of patent applications filed at year t by all sellers (buyers) of firm i. Therefore, $|\text{Sellers' Patents}|_{i,t} \,/\, |\text{Sellers}|_{i,t} \left(\left|\text{Buyers' Patents}\right|_{i,t} \,/\, \left|\text{Buyer}\right|_{i,t}\right)$ captures the average R&D activity of its sellers (buyers). To address potential endogeneity issues, we construct instrumental variables using exposure to foreign patents:

$$\text{IV for } \frac{|\text{Sellers' Patents}|_{i,t}}{|\text{Sellers}|_{i,t}} = \sum_c \frac{|\text{Sellers' Patents}|_{i,c,t-1}}{|\text{Sellers' Patents}|_{i,t-1}} \times \left|\text{Foreign Patents}\right|_{c,t}$$

$$\text{IV for } \frac{\left|\text{Buyers' Patents}\right|_{i,t}}{\left|\text{Buyers}\right|_{i,t}} = \sum_c \frac{\left|\text{Buyers' Patents}\right|_{i,c,t-1}}{\left|\text{Buyers' Patents}\right|_{i,t-1}} \times \left|\text{Foreign Patents}\right|_{c,t}$$

Here, $\left|\text{Foreign Patents}\right|_{c,t-1}$ represents the number of patents in class c filed at time $t-1$ at the Japan Patent Office (JPO) by foreign firms. These indicators are constructed using the triadic patents filed in all of the U.S., Europe, and Japan by excluding Japanese firms' applications. The idea behind using triadic patents is that these patents are likely to capture global technology trends rather than Japan-specific demand-driven inventions. Moreover, by excluding Japanese firms' triadic patents, we can create exogenous shifters for Japanese firms. While Acemoglu et al. (2015) aggregate patent classes into sectors using crosswalks and treat them as exogenous shocks to the sectors, we can directly observe firm-level technology exposures, allowing us to construct the above firm-level instruments. The weights $|\text{Sellers' Patents}|_{i,c,t-1} \,/\, |\text{Sellers' Patents}|_{i,t-1}\left(\left|\text{Buyers' Patents}\right|_{i,c,t-1} \,/\, \left|\text{Buyers' Patents}\right|_{i,t-1}\right)$ capture the buyers' (sellers') "exposure" to each patent class c.

Table 1 shows the estimates from both ordinary least squares (OLS) and instrumental variable (IV) regressions. In both OLS and IV regressions, the coefficients for both average number of seller's and buyer's patent applications are positive and statistically significant.

This suggests that firm's R&D activity is positively affected by the R&D activities of its suppliers and buyers. In the IV estimates presented in columns (4)-(6), the magnitude of the coefficients is larger than in the OLS estimates, suggesting that the OLS estimates may be biased downward.

Table 1: Regression Results

	(1)	(2)	(3)	(4)	(5)	(6)
$\log \frac{\lvert\text{Sellers' Patents}\rvert_{i,t}}{\lvert\text{Sellers}\rvert_{i,t}}$	0.020***		0.019***	0.068***		0.062***
	(0.003)		(0.003)	(0.018)		(0.018)
$\log \frac{\lvert\text{Buyers. Patents}\rvert_{i,t}}{\lvert\text{Buyers}\rvert_{i,t}}$		0.022***	0.020***		0.077***	0.073***
		(0.003)	(0.003)		(0.021)	(0.021)
year fixed effect	Yes	Yes	Yes	Yes	Yes	Yes
firm fixed effect	Yes	Yes	Yes	Yes	Yes	Yes
Fstat				840.4	759.8	373.5
Observations	53,553	53,553	53,553	53,553	53,553	53,553

Standard errors in parentheses

* p<0.10, ** p<0.05, *** p<0.01

Notes: Standard errors are clustered at the industry × year level.

To see that our results are driven by the mechanism related to the production network rather than technological spillovers, Table 2 show that we obtain similar significant results even when we exclude patents that have a citation relationship with the applicant's buyers or sellers. This suggests that our findings is driven by the incentive from production network rather than spillovers through technological class connections.

Table 2: Regression Results (excluding patenting relationships)

	(1)	(2)	(3)	(4)	(5)	(6)
$\log \frac{\lvert\text{Sellers' Patents}\rvert_{i,t}}{\lvert\text{Sellers}\rvert_{i,t}}$	0.018***		0.017***	0.049**		0.043**
	(0.003)		(0.003)	(0.019)		(0.019)
$\log \frac{\lvert\text{Buyers. Patents}\rvert_{i,t}}{\lvert\text{Buyers}\rvert_{i,t}}$		0.020***	0.019***		0.081***	0.079***
		(0.003)	(0.003)		(0.023)	(0.023)
year fixed effect	Yes	Yes	Yes	Yes	Yes	Yes
firm fixed effect	Yes	Yes	Yes	Yes	Yes	Yes
Fstat				805.3	690.5	339.0
Observations	51,338	51,338	51,338	51,338	51,338	51,338

Standard errors in parentheses

* p<0.10, ** p<0.05, *** p<0.01

Notes: Standard errors are clustered at the industry × year level.

In sum, we provides novel evidence on the impact of production networks on firm innovation using a unique integrated database of firm-level transactions and patent data. We find that firm innovation is positively affected by the innovation activities of their suppliers and buyers in the production network, even when controlling for potential endogeneity issues using instrumental variables. Our results suggest that the impact of production networks on innovation goes beyond knowledge spillovers and is driven by the incentives from intermediate goods trade relationships.

3 A Model of Production Networks and R&D

Building on the empirical findings from Section 2, we develop a model that captures the interaction between production networks and innovation. Specifically, we build on Klette and Kortum (2004), a canonical model of innovation and firm dynamics, and introduce a firm-to-firm matching process replicating the age-dependent production network pattern observed in the data. Our original and most critical assumption is that successful R&D can use existing production networks to create and sell new goods, consistent with the positive impact of trading partners' R&D on a firm's own R&D found in the data.

3.1 Settings

A unit measure of infinitely lived households supply a unit measure of production workers, R&D workers, and entrepreneurs.[3]. Households have preferences over a final consumption good, $U_0 = \int_0^\infty \exp(-\rho t) \log Y(t)\, dt$ where $\rho > 0$ is the discount rate and $Y(t)$ is the consumption. The budget constraint is $\dot{A}(t) \leq r(t)A(t) + w(t) + w_H(t) + w_E(t) - P(t)Y(t)$ with the standard no-Ponzi condition, where $A(t)$ is the asset position, $r(t)$ is the interest rate, and $w(t)$, $w_H(t)$, and $w_E(t)$ denote wages for each type of worker. We set the nominal GDP as numeraire. Then, the Euler equation implies $r(t) = \rho$. In what follows, we focus on the stationary equilibrium, and to streamline notation, we drop the time subscripts when they are not confusing.

Products, Firms, and Production Networks

There is a set of intermediate goods, and each intermediate good is denoted by $\omega \in \Omega(t)$. The mass of intermediate goods evolves through the creation of new varieties and exit. Each intermediate good ω is produced by the monopolist although a single monopolist can own multiple product lines and can produce multiple intermediate goods simultaneously. Consider a firm $f \in \mathcal{F}$ that owns product line ω. Let $n(f)$ denote the number of the product lines firm f owns. We drop the firm subscript f from firm variables when it causes no confusion. We denote by $\mathcal{S}(\omega) \subset \Omega$ the set of products that are used as inputs for the product $\omega \in \Omega$. Although the set of buyers can be identified from the inverse mapping of $\mathcal{S}(\cdot)$, for notational simplicity, we denote the subset of buyers of Ω as $\mathcal{B}(\cdot) : \Omega \to \Omega$.

Production Structure Given Networks

Firms use production workers and intermediate inputs for production. Intermediate inputs are imperfect substitutes with a constant elasticity of substitution, $\sigma \geq 1$. Production workers and the composite of intermediate inputs are combined in a Cobb-Douglas aggregator with labor share, β $(0 \leq \beta \leq 1)$:

$$x(\omega) = \frac{1}{\beta^\beta (1-\beta)^{1-\beta}} l(\omega)^\beta \left(\int_{\omega' \in \mathcal{S}(\omega)} x(\omega, \omega')^{\frac{\sigma-1}{\sigma}} d\omega' \right)^{\frac{\sigma}{\sigma-1}(1-\beta)}, \tag{1}$$

[3] The fixed skilled labor abstracts from the underinvestment that happens when other types of workers can also perform R&D activities or when output can be converted to R&D. The recent paper investigating R&D misallocation (Aghion et al. (2023); Liu and Ma (2023)) also use this assumption.

where $l(\omega)$ is demand for production workers used to produce ω; $x(\omega', \omega) = (1-\beta)\, p(\omega)^{-\sigma} c(\omega')^{\sigma} x(\omega')$ is the demand for product ω used to produce ω'; $x(\omega)$ is the total quantity of product ω.

A representative household has a CES utility function with an elasticity of substitution σ, which is the same elasticity as that for production:

$$Y = \left(\int_{\omega \in \Omega} y(\omega)^{\frac{\sigma-1}{\sigma}} d\omega \right)^{\frac{\sigma}{\sigma-1}}, \tag{2}$$

where $y(\omega) = Y \left(\frac{p(\omega)}{P} \right)^{-\sigma}$ is the final demand for product ω. For all ω, $x(\omega)$ satisfies the following market clearing condition:

$$x(\omega) = \int_{\omega' \in \mathcal{B}(\omega)} x(\omega', \omega) d\omega' + y(\omega), \tag{3}$$

Under monopolistic competition, suppliers charge a constant markup over their marginal cost:

$$p(\omega', \omega) = p^{y}(\omega') = \mu c(\omega'), \tag{4}$$

where $p(\omega', \omega)$ is the product price that supplier ω' charges to firm ω, $p^{y}(\omega')$ is the price that supplier ω' charges to the final good consumer, and $\mu = \frac{\sigma}{\sigma-1}$.

R&D, Matching, Entry, and Exit

Firms conduct R&D to acquire product lines of new variety. [4] We make the following assumptions about the set of production networks of sellers and buyers for which new products are available.

Assumption 1. *For a new product line ω' derived from the existing firm's product line ω,*

$$\mathcal{S}(\omega') = \mathcal{S}(\omega),$$

$$\mathcal{B}(\omega') = \mathcal{B}(\omega)$$

In words, existing firms can develop new products and sell them to existing buyers by using intermediate goods from suppliers of each product line owned by the existing product line.

[4] Our model is based on Klette and Kortum (2004), but abstracts from productivity differences and growth. So instead of creative destruction, we assume variety creation.

The product level cost function follow Klette and Kortum (2004):

$$\tilde{\phi}(x, n) = nw_H\phi(\lambda), \tag{5}$$

where $\phi(\lambda) = \frac{1}{\varphi}\lambda^\gamma$, λ is a per product innovation rate and $\gamma > 1$. [5]

Firms are matched randomly with other firms at exogenous rates, ζ. The match is exogenously terminated at the rate δ_M or if one of the firms in either side of the match exits. Firms die at an exogenous rate δ_F. Entrepreneur have access to a linear entry technology, where each R&D worker generates a flow of λ_E. Entrants start with ζ_0 mass of randomly chosen sellers or buyers.

3.2 Characterization of decentralized Equilibrium

The model involves the problem of tracking trading relationships between continuous products, which is impossible to solve in general, but we show that under our setting the equilibrium conditions are summarized by the following objective function over the state space of ages, a.

Thereafter, let $F(a)$ be the cumulative density of products with respect to age a, and we will use a instead of ω. Let $N(t)$ be the total measure of products. Let $F(a, t)$ be defined so that the fraction of products which is owned by firms with age less than or equal to a at time t is $F(a, t)/N(t)$. Let $N_f(t)$ denote the total mass of firms. Because the total mass of firms evolves according to $\dot{N}_f(t) = \lambda_E - \delta_F N_f(t)$, in the steady state $N_f = \lambda_E/\delta_F$. We begin with a matching distribution. As we will show later, given that the optimal innovation rate per product, λ, is a function of age, we can show the transition equation that the matching process solves the following differential equation.

Proposition 1. *Law of motion of matching process: The distribution of matched products less than age a' connected with an age a product is given by*

$$\underbrace{\frac{\partial}{\partial a'}m(a'; a) + \frac{\partial}{\partial a}m(a'; a)}_{\text{time evolution of themaching distribution}}$$

[5] This product level cost function can be micro-founded by CRS innovation production function as R&D workers and number of product as inputs. Namely, a firm f hires $l_H(f)$ units of skilled workers to adds one more product at the flow rate $\Lambda(f) = n(f)^{1-1/\gamma}\left(\varphi l_H(f)\right)^{1/\gamma}$,where Λ is firm level flow rate.

$$= \underbrace{-(\delta+\delta_N)\,m(a';a)}_{\text{link destruction}} + \underbrace{\frac{\zeta}{N_f}f(a')}_{\text{random matching}} + \underbrace{\underbrace{\lambda\,(a')}_{\text{innovation rate}}\quad m(a';a) + \frac{\zeta_0}{N_f}\lambda_E\boldsymbol{\delta}(a)}_{\text{new product made by parnters with} a'}, \tag{6}$$

where $\lambda\,(a)$ is the innovation rate of age a firm, subject to the boundary conditions

$$m(a';0) = \frac{\zeta_0}{N_f}f(a') \tag{7}$$

The matching distribution evolves over time because of the following reasons. First, the matched product is added when the currently linked firm creates a new variety at rate $\lambda\,(a')$. Second, the matched product is lost when the link is terminated at the exogenous rate δ or when the linked firm exits at the exogenous rate δ_N. Finally, the matched product is added when there is random matching with new firms.

In general, if connections change stochastically, one needs to track changes in connections among countless firms each period. However, from the perspective of connected partners, the randomness disappears due to the law of large numbers. Thus, the distribution can be summarized by age. This formulation opens new possibilities for modeling two-sided production network structures.

Next, armed with a matching distribution as a function of age, we can characterize the equilibrium objects of firms using the demand shifters $D(a)$ and the cost shifters $c(a)$. These functions solve the two fixed point problems given by the following Lemma.

Lemma 1. *The demand shifters and cost shifters solve the following fixed points:*

$$D(a) = (1-\beta)\,\mu^{-\sigma}\int\left(\frac{c(a')}{w}\right)^{\frac{\beta}{1-\beta}(\sigma-1)} D(a')m(a';a)da' + \underbrace{PY}_{GDP} \tag{8}$$

$$c\,(a) = w^{\beta}\left(\int\left(\mu c(a')\right)^{1-\sigma} m(a';a)da'\right)^{\frac{1-\beta}{1-\sigma}} \tag{9}$$

where $P = \mu\left(\int c(a)^{1-\sigma}dF(a)\right)^{\frac{1}{1-\sigma}}$.

The demand shifter represents the relative size of demand faced by firms at age a relative to the size of the economy, PY, as measured by nominal GDP. The size of demand for intermediate goods is the sum of the portion of aggregate demand $D(a')$ coming from different ages toward a for all a'. The presence of the markup in equation (8) implies that the demand for intermediate goods shrinks as double marginalization is repeated for each intermediate goods transaction. The cost shifters (9) are standard, but the point is that the matching function is a function of a, so the cost function can also be summarized by a.

Using this Lemma 1, it is easy to recover the product level equilibrium objects as a function of a. The profit generated by a product with age a can be characterized by

$$\pi(a) = \left(1 - \frac{1}{\mu}\right)\left(\frac{\mu c(a)}{P}\right)^{1-\sigma} D(a)$$

The fact that profit is a function of age suggests that the firm's optimization problem can also be summarized in terms of age. A firm with n product lines and age a maximizes the value $V^F(n, a)$

$$\begin{aligned} r(t)V^F(n,a) = {} & \underbrace{n\pi(a)}_{\text{profits}} - \underbrace{\delta_N n\left\{V^F(n,a) - V^F(n-1,a)\right\}}_{\text{product exit}} + \underbrace{V_a^F(n,a)}_{\text{age effect}} \\ & + \max_{\lambda \geq 0}\left[\underbrace{n\lambda\left\{V^F(n+1,a) - V^F(n,a)\right\}}_{\text{Expansion of variety}} - \underbrace{n w_H \phi(\lambda)}_{\text{R\&D costs}}\right]. \end{aligned} \tag{10}$$

In words, the first term on the right-hand side is the total static profit. The second term is the change in firm value due to the exogenous withdrawal of one of its product lines. The third term is the change in firm value due to aging. The fourth term is the change in firm value if a product is added when a new product line arrives with a Poisson arrival rate $n\lambda$. The last term is the R&D cost.

We can show that the value of each firm can be expressed as the sum of the value of the product lines, defined as the net present discounted value of profits from a product line. To show this, guess

$$V^F(n,a) = nV(a)$$

where $V(a)$ is the value of product lines owned by an age a firm, and obtain the following HJB equation for $V(a)$:

$$(\rho + \delta_N) V(a) = \pi(a) + V_a(a) + \max_{\lambda \geq 0}\left[\lambda V(a) - w_H \phi(\lambda)\right] \tag{11}$$

Finally, the first order condition yields an optimal innovation rate:

$$\lambda(a) = \left\{ \frac{\overline{\phi}}{\gamma w_H} V(a) \right\}^{\frac{1}{\gamma - 1}}, \tag{12}$$

which confirms λ is also a function of age.

Kolmogorov Forward Equations

Standard arguments establish that the differential equation governing the evolution of the product density, $f(a)$ at steady state takes the following form:

$$0 = -\frac{\partial f(a)}{\partial a} + (\lambda(a) - \delta_F) f(a) + \lambda_E \boldsymbol{\delta}(a) \tag{13}$$

where $\boldsymbol{\delta}(a)$ denotes the Dirac delta function, which is zero everywhere except if $a = 0$, and satisfies $\int \boldsymbol{\delta}(a) da = 1$. The time evolution of the firm distribution $\frac{\partial f(a)}{\partial t} = 0$ is consistent with the sum of the four terms in the right-hand side: The first term captures the increase in firm age, the second term captures the increase in the number of product lines due to the successful R&D by age a firm minus the exogenous exit of product lines, and the third term captures the addition of product lines due to the entry of age 0 firm.

Labor Market Clearing Conditions

The labor market clearing condition can be expressed using the demand shifters,

$$w = \beta \mu^{-\sigma} \int \left(\frac{c(a)}{P} \right)^{1-\sigma} D(a) f(a) da \tag{14}$$

The high skilled labor market clearing condition is

$$L_H = \int \phi(\lambda(a)) f(a) da \tag{15}$$

4 R&D Misallocation

The pattern of inter-firm production network linkages replicated by the model has direct implications for the allocative efficiency of R&D. To investigate the inefficiencies in the allocation of R&D workers in the decentralized equilibrium, we characterize the allocation of the social planner's R&D workers and compare it to the allocation in the decentralized equilibrium.

4.1 Allocation of R&D workers in the decentralized equilibrium

We begin by characterizing the allocations in the decentralized equilibrium.

Proposition 2. *The allocation of R&D workers in the decentralized equilibrium:*

$$
\begin{aligned}
l_H(a) &= \frac{V(a)^{\frac{\gamma}{\gamma-1}}}{\int V(a)^{\frac{\gamma}{\gamma-1}} f(a)da} \\
&\propto V(a)^{\frac{\gamma}{\gamma-1}}
\end{aligned}
$$

Product Value function $V(a)$ in decentralized equilibrium solves

$$
\begin{aligned}
\rho V(a) = & \underbrace{\frac{1}{\sigma} r(a)}_{profit} \underbrace{-\delta_N V(a)}_{product\ death} + \underbrace{V_a(a)}_{age\ effect} \\
& + \underbrace{\left[\lambda(a) V(a) - w_H \phi(\lambda(a))\right]}_{net\ value\ of\ innovation}
\end{aligned}
$$

where revenue is $r(a) = \left(\frac{c(a)}{P}\right)^{1-\sigma} D(a)$

This Proposition implies that the allocation of R&D workers in the decentralized equilibrium is proportional to the$\frac{\gamma}{\gamma-1}$ power of the product value function. γ is the curvature of the R&D cost function, reflecting the fact that the more R&D workers are hired, the more their marginal cost increases. As the two fixed points (8) and (9) in Proposition () implies, $V(a)$ is the discounted present value of the profit determined from the production network, so we can see that distortions in the goods market (in the sense of ex-ante expectations) also directly affect the allocation of R&D workers. To explore this point, we introduce the social planner problem in the next section and compare it to the allocation in this decentralized equilibrium.

4.2 A social planner's problem

The social planner maximizes the discounted utility flow

$$U_0 = \int_0^{\infty} \exp(-\rho t) \log Y(t)\, dt$$

where $Y = \left(\int y(a)^{\frac{\sigma-1}{\sigma}} dF(a)\right)^{\frac{\sigma}{\sigma-1}}$ subject to the following constraints: (1), (3), (6), (14), (15), and (13). We use the fact that the allocation of production workers is static to solve a two-stage maximization problem. First, we characterize the allocation of general workers with $f(a)$ and $m(a';a)$ as given. Next, this allocation is solved as a dynamic optimization problem using a maximum value function with respect to $f(a)$ and $m(a';a)$.

First, the social planner's static allocation $(y(a), x(a), l(a), x(a', a))$ problem satisfies the following problems:

$$\max\left(\int y(a)^{\frac{\sigma-1}{\sigma}} dF(a)\right)^{\frac{\sigma}{\sigma-1}}$$

subject to

$$x(a) = \frac{1}{\beta^{\beta}(1-\beta)^{1-\beta}} l(a)^{\beta}\left(\int x(a', a)^{\frac{\sigma-1}{\sigma}} m(a';a)da'\right)^{\frac{\sigma}{\sigma-1}(1-\beta)}, \tag{16}$$

$$x(a) = \int x(a', a)m(a', a)da' + y(a), \tag{17}$$

$$\int l(a)dF(a) = 1 \tag{18}$$

We make the following transformation to compare this problem with the characterization in the cost function in the decentralized equilibrium. By relabeling the Lagrange multipliers of the above problem appropriately, we obtain the following Lemma.

Lemma 2. *The static allocation can be characterized by the following P^{SP}, $D(a)^{SP}$ and $c(a)^{SP}$. They solve the following equations.*

$$D^{SP}(a) = (1-\beta)\int \left[c(a')^{SP}\right]^{\frac{\beta}{1-\beta}(\sigma-1)} D^{SP}(a')m(a';a)da' + 1, \tag{19}$$

$$c^{SP}(a) = \left(\int \left(c^{SP}(a') \right)^{1-\sigma} m(a'; a) da' \right)^{\frac{1-\beta}{1-\sigma}}, \tag{20}$$

$$\left(P^{SP} \right)^{1-\sigma} = \beta \int c^{SP}(a)^{1-\sigma} D^{SP}(a) f(a) da, \tag{21}$$

$$P^{SP} = \left(\int c^{SP}(a)^{1-\sigma} f(a) da \right)^{\frac{1}{1-\sigma}} \tag{22}$$

Given the solutions, we could recover the solution of the original problem ($x^{SP}(a), y^{SP}(a)$, and $l^{SP}(a)$) as $x^{SP}(a) = \frac{D^{SP}(a)}{P^{1-\sigma} c^{SP}(a)^{\sigma}}$, $y^{SP}(a) = c^{SP}(a)^{-\sigma}$, and $l^{SP}(a) = \beta c^{SP}(a) x^{SP}(a)$.

This Lemma 2 is useful for characterizing static allocations because it gives the social planner analogs of the demand (28) and cost shifters (9) and the price index in the decentralized equilibrium. Also, by comparing with Lemma 1, we can see that the two allocations coincide only when $\mu = 1$, i.e., the goods market equilibrium is inefficient.

We then redefine the social planner's problem using Lemma 2: using $y^{SP}(a) = c^{SP}(a)^{-\sigma}$ and (21) (22), we transform the social planner's objective function as follows

$$\begin{aligned} U_0 &= \int_0^{\infty} \exp(-\rho t) \log \left(\int c(a)^{1-\sigma} f(a) da \right) dt \\ &= \int_0^{\infty} \exp(-\rho t) \frac{\sigma - 1}{\sigma} \log \left(\beta \int c^{SP}(a)^{1-\sigma} D^{SP}(a) f(a) da \right) dt \end{aligned}$$

The social planner maximizes the above objective function subject to (13), (6), (19), (20), (21), and (22). The following Proposition characterizes the optimal allocation of the social planner's R&D workers.

Proposition 3. *A social planner's allocation of R&D workers:*

$$l_H^{SP}(a) \propto \left(V(a)^{SP} \right)^{\frac{\gamma}{\gamma-1}},$$

where $V(a)^{SP}$, μ_λ and $\mu_m(a, a')$ are shadow values for (13), (21) and (6), respectively.

A social value function:

Social value function $V^{SP}(a)$ solves

$$\begin{aligned} \rho V^{SP}(a) =& r^{SP}(a) - \delta_N V^{SP}(a) + V_a^{SP}(a) \\ &+ \left[\lambda^{SP}(a) V^{SP}(a) - w_H \phi(\lambda^{SP}(a)) \right] \end{aligned}$$

$$+\int V^M(a',a)\left\{\frac{\zeta}{N_f}f(a')-\delta_M m(a',a)\right\}da'$$

and where value of matching $V^M(a',a)$ solves

$$\begin{aligned}(\rho+\delta_F+\delta_M-\lambda(a))\,V^M(a',a)=&V_a^M(a',a)+V_{a'}^M(a',a)\\&+\Omega(a',a)r^{SP}(a)\end{aligned}\tag{23}$$

where where $V(a)^{SP}$, μ_λ and $V^M(a',a)$ are shadow values for (13), (21) and (6), respectively. And $\Omega(a',a)=\frac{c(a')^{1-\sigma}}{\int c(a'')^{1-\sigma}m(a'',a')da''}$ is an expenditure share of age a over all intermediate consumption of age a'

Similar to the lemma in 2, we find that the allocation of R&D workers in the social planner's solution and decentralized equilibrium is proportional to the number of values. In other words, to know the allocation of R&D workers, we focus on the value functions. The following proposition characterizes the value functions of the decentralized equilibrium and the social planner solution. By comparing the value functions of Propositions 2 and 3, we can see that there are two differences. These are explained separately below.

Misallocation from intensive margin of the network

The first channel of misallocation is related to the networks' intensive margin. Firms' revenues in decentralized equilibrium are characterized from fixed points based on network connection patterns and determined from markups, as can be seen from Lemma 1. On the other hand, the revenue analog in the social planner is also network-based, but as we will see in 2, the solution for the social planner is given by solving a fixed point problem that does not involve markups. Although all firms (and their products) in our model have a common markup, the heterogeneity of the network at each age distorts the goods market due to *a heterogeneous degree of double marginalization*. This force is closely related to the literature on the misallocation of goods markets in networks (Liu (2019); Baqaee and Farhi (2020); Bigio and La'O (2020); Baqaee et al. (2023); Osotimehin and Popov (2023)), where firms are exposed to substantially more distortions by trading with distorting downstream firms, even if those firms themselves do not impose many markups.

Misallocation from extensive margin of the network

The second channel, related to the extensive margin, is new in the literature.

$$\int V^M(a', a)\left\{\frac{\zeta}{N_f}f(a') - \delta_M m(a', a)\right\}da'$$

in the planner's social value function is a term not present in the decentralized equilibrium. $V_m(a, a')$ denotes the discounted present value of the revenue that a' would obtain by trading with a. This is evident in the second term of equation (23). $\Omega(a', a)(1-\beta)\, r^{SP}(a)$ represents how much of revenue(=costs) of a attributable to intermediate goods $(1-\beta)$ and $\Omega(a', a)$ is expenditure share of a' of a over the intermediate inputs. The total social value is calculated by multiplying the discounted present value of this link by the number of net links generated ($= \frac{\zeta}{N_f}f(a') - \delta_M m(a', a)$) and summing this for all ages a'. This effect is not internalized in the decentralized equilibrium and grows with the number of connections. The direction of this effect is ambiguous, but as we will see later, the former prevails in our calibrations, and this effect also acts as an effect of underinvestment by older firms. Appendix D gives a complete characterization of the social planner problem.

5 Quantitative Exercise

We solve the model numerically, confirm that it replicates the network and R&D age dependence observed in the data, and discuss its properties. We then compare the allocation of R&D workers in the decentralized equilibrium with that in the social planner solution.

5.1 Numerical example

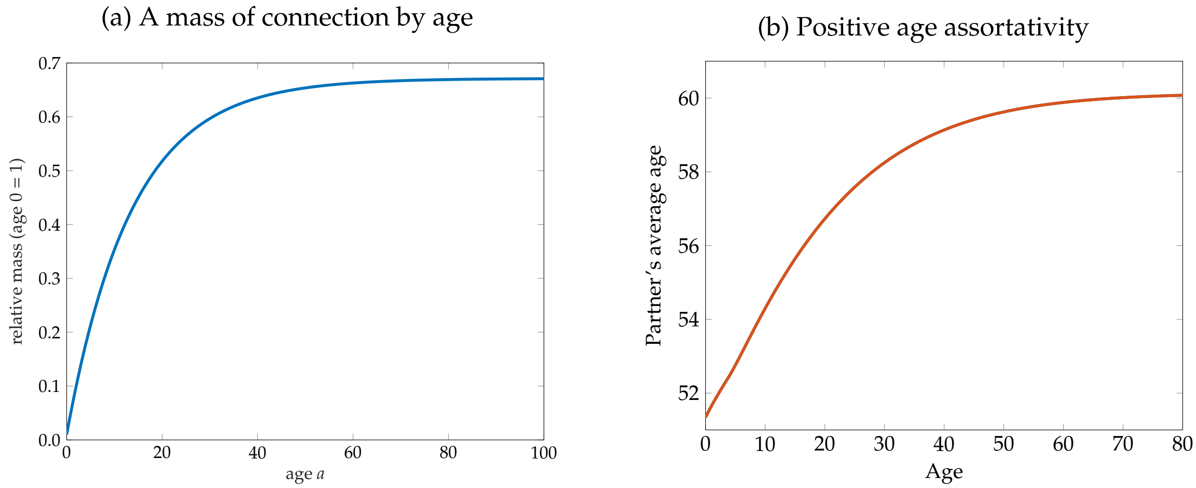
The following parameters were calibrated according to previous studies: the elasticity of substitution parameter σ was set to 4, based on Arkolakis et al. (2023). The matching rate, ζ, was set to 0.2 according to Arkolakis et al. (2023). For the exogenous link and product mortality, we set it at 0.1, following Peters and Walsh (2022). In a future version of this paper, all parameters will be structurally estimated using the moments of the Japanese economy.

Parameter	Description	Value
ρ	Discount Rate	0.05
σ	Elasticity of Substitution	4
β	Labor Cost Share related parameter	0.3
γ	Curvature of R&D Cost	2
ζ	Matching Drift	0.2
δ	Link Death Rate	0.1
δ_N	Product Death Rate	0.1
$\overline{\phi}$	Relative Efficiency of R&D by Incumbent	0.01

5.2 Simulation Results

In this section, we examine the properties of the calibrated model. First, we confirm that the age-dependent production network pattern is replicated as in the data. In Figure 5a, we plot the size of the connection with the partner. The results show that both of these values increase monotonically, which is consistent with the data. On the other hand, 5b, we take the age of the firm on the x-axis and the average age of the connected partners on the y-axis. As we saw in Section 2, as the age of the firm increases, the age of the connected partner also increases, so the steady-state distribution is more likely to be matched with a partner close to its own age. This replicates positive age assortativity. Our approach of using a dynamic structure, simple random matching, and link stickiness to replicate assortativity among firms is a novel approach in the literature. For example, Arkolakis et al. (2023) use static two-sided matching to reproduce assortative matching in terms of productivity, while Demir et al. (2024) model positive assortativity among high-productivity firms with high wage premiums using a static directed search model. In contrast, our model captures the dynamic evolution of the production network and the resulting age-dependent patterns observed in the data.

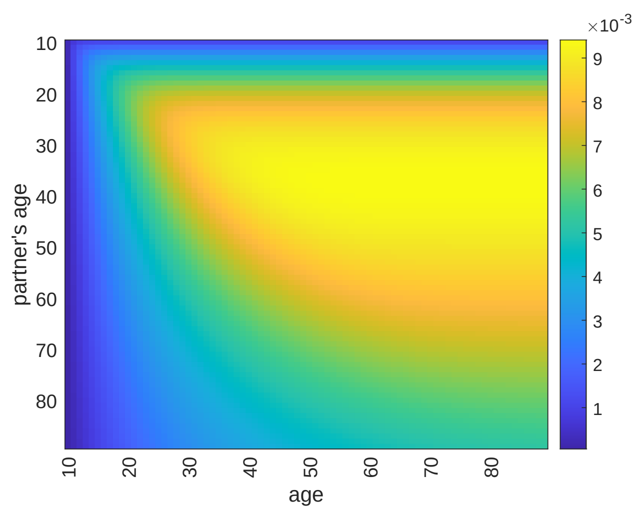
Figure 5: Age-dependent production networks in the model



Notes: This figure plots the firm age on the x-axis, and the average age of the connection masses and their counterparts on the y-axis, respectively, in equilibrium with the model.

Figure 6 depicts these two age dependencies simultaneously using a heat map. As age increases, the density of connections increases and the distribution of partners with connections is skewed toward older age groups.

Figure 6: A heat-map of the matching density



Notes: This figure plots firm age on the x-axis and the average age of connected firms on the y-axis in the equilibrium of the model.

5.3 Age-dependent production networks and R&D misallocation

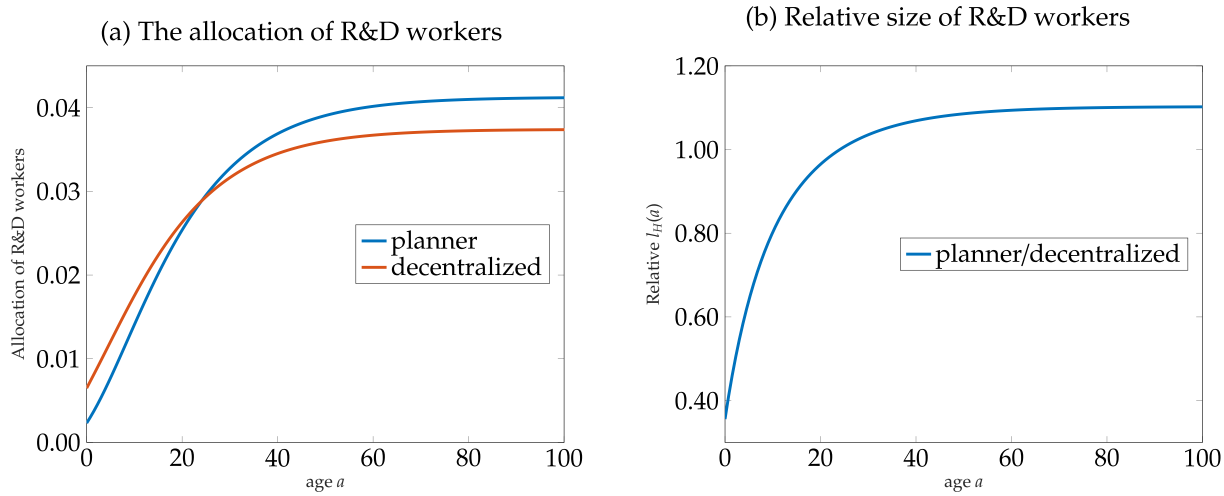
The inter-firm production network linkage pattern replicated by the model has a direct impact on R&D allocative efficiency. We restate the analytical representation of R&D worker allocation in Lemma 2 and 3.

$$
\begin{aligned}
l_H(a) &\propto V(a)^{\frac{\gamma}{\gamma-1}} \\
l_H^{SP}(a) &\propto \left(V(a)^{SP}\right)^{\frac{\gamma}{\gamma-1}}
\end{aligned}
$$

From this, we see two main differences in the allocation of R&D workers in the social planner's solution and the decentralized equilibrium: The first channel is a double marginalization channel of different magnitude, related to the intensive margin. In a network economy, the size of a firm's sales is determined by the final and intermediate demand of other firms. The fact that older firms are more interconnected and that their counterparts are also older firms suggests that firms' expected returns are too low due to the high degree of double marginalization. Since firms are incentivized to conduct R&D by sales-based profits, this leads to the conclusion that the amount of R&D is too small relative to the optimal allocation.

The second channel is the effect arising from differences in extensive margins, which are not present in the decentralized equilibrium. $V^M(a', a)$ represents the shadow value attributed to the product with a' when the product with a is connected. The sum of the connection weights of the connection partners of each a' represents the sum of the values attributed to the existing connection partners when a new link is created. In a decentralized equilibrium, this effect is not internalized, and this effect increases with the number of connections. In our model, the older a firm is, the more connections it has, and the size of the value is disproportionately larger for older firms. Figure 7 compares the simulation results of the two allocations. By the mechanism described above, R&D workers are under-allocated to older firms.

Figure 7: R&D misallocation.



Notes: This figure plots the firm age on the x-axis, the mass of connected firms and the average intermediate goods input share on the y-axis in the BGP equilibrium of the model.

Since both of these effects lead to underinvestment by firms with more network connections, the decentralized equilibrium is inefficient. The optimal allocation implied by this calibration has a total output 20% larger than the decentralized equilibrium.

6 Conclusion

In conclusion, our study sheds light on the interaction between production networks and R&D activities by constructing a unique dataset that combines firm-level production network and patent data in Japan. The dynamic model we propose explains firms' R&D decisions, which are affected by their position in the production network, and their misallocation of R&D workers. Our model successfully reproduces the production network structure observed in the data, particularly the age-dependent production network pattern. The results suggest that older firms tend to underinvest in R&D relative to the optimal allocation. This study provides new insights into the dynamic and endogenous relationship between production networks and R&D, with implications for more efficient allocation of economic resources. A revised version of the paper currently in progress will estimate the structure of the model using Japanese data.

References

Acemoglu, D., U. Akcigit, H. Alp, N. Bloom, and W. Kerr (2018, November). Innovation, reallocation, and growth. *The American economic review 108*(11), 3450–3491.

Acemoglu, D., U. Akcigit, and W. R. Kerr (2015). *Networks and the Macroeconomy: An Empirical Exploration*. Harvard Business School.

Acemoglu, D. and P. D. Azar (2020). Endogenous production networks. *Econometrica: journal of the Econometric Society 88*(1), 33–82.

Acemoglu, D. and V. M. Carvalho (2012). The network origins of aggregate fluctuations. *Econometrica: journal of the Econometric Society 80*(5), 1977–2016.

Aekka, A. and K. Gaurav (2023). Endogenous production networks and firm dynamics.

Aghion, P., A. B. T. Boppart, P. J. Klenow, and H. Li (2023). Good rents versus bad rents: R&D misallocation and growth.

Arkolakis, C., F. Huneeus, and Y. Miyauchi (2023, February). Spatial production networks.

Ayerst, S. (2023). Innovator heterogeneity, R&D misallocation and the productivity growth slowdown.

Bai, X., Y. Li, and F. Zanetti (2023). The causal effects of global supply chain disruption on macroeconomic outcomes: Theory and evidence. Technical report, Mimeo.

Baqaee, D., A. Burstein, C. Duprez, and E. Farhi (2023). *Supplier Churn and Growth: A Micro-to-Macro Analysis*. National Bureau of Economic Research.

Baqaee, D. R. and E. Farhi (2020, February). Productivity and misallocation in general equilibrium. *The quarterly journal of economics 135*(1), 105–163.

Bernard, A. B., E. Dhyne, G. Magerman, K. Manova, and A. Moxnes (2019, January). The origins of firm heterogeneity: A production network approach.

Bernard, A. B., E. Dhyne, G. Magerman, K. Manova, and A. Moxnes (2022, July). The origins of firm heterogeneity: A production network approach. *The journal of political economy 130*(7), 1765–1804.

Bigio, S. and J. La'O (2020, May). Distortions in production networks*. *The quarterly journal of economics 135*(4), 2187–2253.

Boehm, C. E., A. Flaaen, and N. Pandalai-Nayar (2019, March). Input linkages and the transmission of shocks: Firm-level evidence from the 2011 tōhoku earthquake. *The review of economics and statistics 101*(1), 60–75.

Carvalho, V. M., M. Nirei, Y. U. Saito, and A. Tahbaz-Salehi (2020, December). Supply chain disruptions: Evidence from the great east japan earthquake*. *The quarterly journal of economics 136*(2), 1255–1321.

Daisuke, F. (2017). The dynamics of inter-firm networks and firm growth.

Demir, B., A. C. Fieler, D. Y. Xu, and K. K. Yang (2024, January). O-Ring production networks. *The journal of political economy 132*(1), 200–247.

Eaton, J., S. S. Kortum, and F. Kramarz (2022, January). Firm-to-Firm trade: Imports, exports, and the labor market.

Huneeus, F. (2018). Production network dynamics and the propagation of shocks. *Princeton University*.

Klette, T. J. and S. Kortum (2004). Innovating firms and aggregate innovation. *The journal of political economy 112*(5), 986–1018.

Liu, E. (2019, August). Industrial policies in production networks*. *The quarterly journal of economics 134*(4), 1883–1948.

Liu, E. and S. Ma (2023, December). Innovation networks and R&D allocation.

Oberfield, E. (2018). A theory of input-output architecture. *Econometrica: journal of the Econometric Society 86*(2), 559–589.

Osotimehin, S. and L. Popov (2023, January). Misallocation and intersectoral linkages. *Review of economic dynamics*.

Peters, M. and C. Walsh (2022). Population growth and firm-product dynamics.

Romer, P. M. (1990). Endogenous technological change. *The journal of political economy 98*(5), S71–S102.

Appendix for "Production Networks and R&D Misallocation"

A Additional Empirical results

A.1 Cross-sectional correlation between production network and citation network.

In this section, we report that the two overlap at the firm level as well: using data from the combined TDB and IIP patent databases, we report that citation relationships in Japan are associated with production networks. Conditional on transactional relationships, the probability of the existence of a citation relationship is about 30 times greater than that of a random benchmark contrast. In addition, the probability decreases monotonically with network distance. This indicates the importance of distinguishing between the two type of networks using firm-level data.

The integrated database contains a universe of Japanese firms that have filed at least one patent application between 1998 and 2019. First, a patent citation matrix was created with the cited firms in columns and the cited firms in rows, where 1 represents the presence of a citation relationship and 0 represents the absence of one. Similarly, an Input-Output matrix was constructed to represent the inter-firm relationships within the patent firms. In this matrix, the columns represent suppliers and the rows represent purchasers, with 1 representing the presence or absence of a business relationship and 0 representing the absence of a business relationship. The elements of these two network matrices represents possible connections between Japanese patent firms-to-firm networks, and based on this, we investigate the relationship between production networks and patent networks.

Table A1: Summary of Production and Patent Network Links

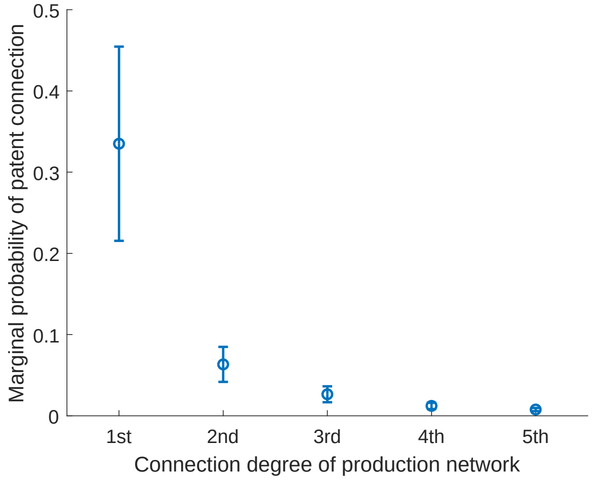
Number of Patent Firms	10,481	
Possible Combinations of Firms	109,851,361	
Active Patent Filing Links	2,548,881	(2.3%)
Active Production Network Links	1,647,770	(1.5%)

The left column of Figure (a) uses a linear probability model to represent the relationship between direct inter-firm relationships and patent citation relationships; columns 1 through 4 measure the impact of inter-firm transactions on patent citations by applying different control variables. When there is a direct business relationship between firms, the probability of a patent citation relationship increases by about 35% compared to the random benchmark.

Next, the right column of Figure (a) shows how higher-order linkages in the production network are related to the patent citation relationship. The probability of patent citation decreases gradually with higher-order linkages, such as second- and third-order links, and becomes almost irrelevant for fourth-order and higher linkages. Finally, (b) visually depicts the decay of the probability of a patent citation relationship with the order of the linkages in the production network.

Table A2: Production Network and Patent connection

(a) Network degrees and patent connection



(b) Network Connection and Patent cintation relationship

	First Degree Connection				Higher Order Degree Connection			
	(1)	(2)	(3)	(4)	Second	Third	Fourth	Fifth
network connection	0.343***	0.339***	0.344***	0.335***	0.063***	0.026***	0.012***	0.007***
	(0.062)	(0.062)	(0.062)	(0.061)	(0.011)	(0.005)	(0.002)	(0.001)
Industry Pair FE	✓			✓	✓	✓	✓	✓
Size Pair FE		✓		✓	✓	✓	✓	✓
Prefecture Pair FE			✓	✓	✓	✓	✓	✓
Observations	93161103	93161104	93161104	93161103	93161103	93161103	93161103	93161103

Standard errors in parentheses

* $p<0.10$, ** $p<0.05$, *** $p<0.01$

Notes: Table 2b regresses the possible combinations of patent firm-to-firm networks in Table 1 on the linkage between production and patent networks and examines the correlation between the two. As a control term, we control for firm-to-firm characteristics. These pairs include (a) the two-digit Japanese Standard Industrial Classification, (b) firm deciles, and (c) prefectures as administrative units in Japan. Table a plots these coefficients.

Alternative measurement

To address the concern about the parametric assumption of linear probability model, we also report the conditional probability given there is a connection in production network by different subset of the samples, $g \in G$, $P_C(\text{patent connection} = 1|\text{network connection} = 1, G)$.

The empirical conditional probability will be computed by

$$\hat{P}_C = \frac{|\text{patent connection} = 1 \cap \text{network connection} = 1, \text{g}|}{|\text{network connection} = 1, \text{g}|},$$

We plot them against unconditional probability, $P_U(\text{patent connection} = 1, G)$ computed by

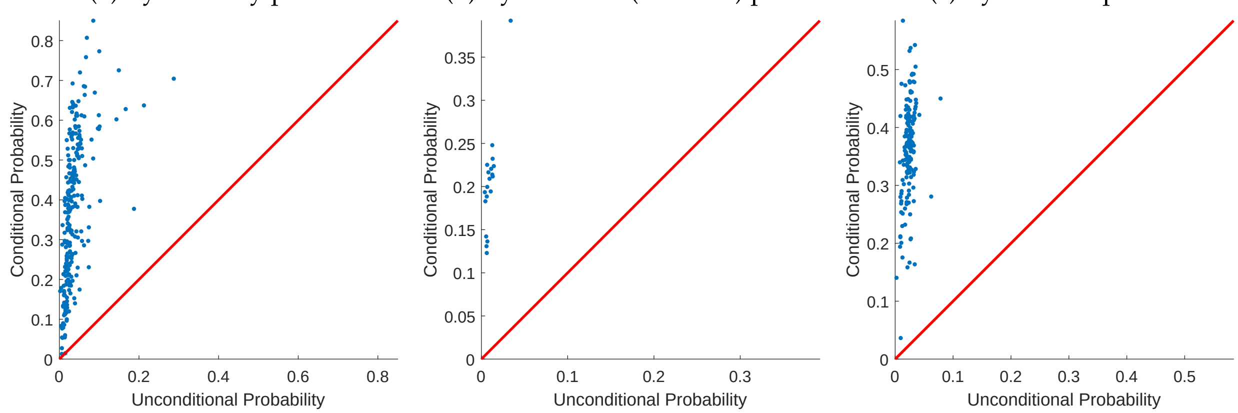
$$\hat{P}_U = \frac{|\text{patent connection} = 1, g|}{|\text{g}|},$$

We choose the subset of the sample by every possible combination of pair of firm's characteristics, such as industry, firm size and locations. Figure A1 reports the conditional and unconditional probabilities that there is a citation relationship, computed from a subset of the sample for all possible combinations of firms in the group for industry, firm size, and location.

For both combinations, the unconditional probability is close to zero because the matrices are very sparse, whereas the average probability of a citation relationship is 30-40%, given the production network connections.

Figure A1: Conditional and unconditional probability of patent connection.

(a) By industry pairs (b) By firm size (decitiles) pairs (c) By location pairs



Notes: The x-axis represents the unconditional probability $\frac{|\text{patent connection} = 1, g|}{|\text{g}|}$, and the y-axis depicts the conditional probability $\frac{|\text{patent connection} = 1 \cap \text{network connection} = 1, \text{g}|}{|\text{network connection} = 1, \text{g}|}$. The groups g comprise all possible combinations of pairs of firm's characteristics. These pairs include (a) two-digit Japan Standard Industrial Classification, (b) decile of the firm, and (c) prefectures in Japan as administrative units. Groups with fewer than 50 samples in either patent relationships or production network connections are excluded from the samples.

B Equation list of Equilibrium

The list of equations for solving the steady state equilibrium is as follows: (i) the value of product $V(a)$; (ii) the innovation rates $\lambda(a)$; (iii) the distribution of products $f(a)$; (vi) the distribution of matched buyers and suppliers $m(a';a)$; (v) the cost function $c(a)$; (vi) the demand shifter $D(a)$; (vii) the wage for production worker w; (viii) the wage for R&D worker w_H, and (ix) the price index P

1. The value of products:

$$(\rho+\delta_F-\lambda(a))\,V(a)=\left(1-\frac{1}{\mu}\right)r(a)+V_a(a)-w_H\phi(\lambda(a))$$

   where revenue, $r(a)$ is

$$r(a)=\left(\frac{\mu c(a)}{P}\right)^{1-\sigma}D(a)$$

   and the price index, P is

$$P=\mu\left(\int c(a)^{1-\sigma}dF(a)\right)^{\frac{1}{1-\sigma}}$$

2. FOC for the innovation rate:

$$\lambda(a)=\left\{\frac{\overline{\phi}}{\gamma w_H}V(a)\right\}^{\frac{1}{\gamma-1}} \tag{24}$$

3. The distribution of products $f(a)$:

$$0=-\frac{\partial}{\partial a}f(a)+(\lambda(a)-\delta_F)\,f(a)+\lambda_E\boldsymbol{\delta}(a) \tag{25}$$

4. The distributions of matched products $m(a';a)$:

$$\begin{aligned}\frac{\partial}{\partial a}m(a';a)=&-\frac{\partial}{\partial a'}m(a';a)-(\delta_M+\delta_F)\,m(a';a)+\frac{\zeta}{N_f}f(a')\\&+\zeta_0\frac{\lambda_E}{N_f}\boldsymbol{\delta}(a')+\lambda(a')\,m(a';a)\end{aligned} \tag{26}$$

   subject to the boundary condition $m(a';0)=\zeta_0 f(a')$.

5. The cost shifter $c\,(a)$ satisfies

$$c\,(a) = w^{\beta}\left(\int\left(\mu c(a')\right)^{1-\sigma} m(a';a)da'\right)^{\frac{1-\beta}{1-\sigma}} \tag{27}$$

where $w = \frac{\beta}{\mu-(1-\beta)}$.

6. The demand shifter for an age a product $D(a)$ satisfies

$$D(a) = (1-\beta)\,\mu^{-\sigma}\int\left[\frac{c(a')}{w}\right]^{\frac{\beta}{1-\beta}(\sigma-1)} D(a')m(a';a)da' + 1 \tag{28}$$

7. The skilled wage w_H satisfies the skilled labor market clearing condition

$$1 = \int \phi\left(\lambda(a)\right) f(a)da \tag{29}$$

Equation list of social planner solution:

1. Social value function

$$(\rho+\delta_F-\lambda(a))\,V^{SP}(a) = r^{SP}(a) + \frac{\partial}{\partial a}V^{SP}(a) - w_H^{SP}\phi\left(\lambda(a)\right) + \int V^M\left(a,a'\right)\left\{\frac{\zeta}{N_f} - \delta_M\frac{m(a,a')}{f(a)}\right\}da',$$

where

$$r^{SP}(a) = \left(\frac{c^{SP}(a)}{P^{SP}}\right)^{1-\sigma} D^{SP}(a)$$

$$P^{SP} = \left(\int c^{SP}(a)^{1-\sigma} f(a)da\right)^{\frac{1}{1-\sigma}}$$

$$\begin{aligned}(\rho+\delta_F+\delta_M-\lambda\,(a))\,V^M\left(a,a'\right) = {} & \frac{\partial}{\partial a}V^M\left(a,a'\right) + \frac{\partial}{\partial a'}V^M\left(a,a'\right) \\ & + r^{SP}(a)\left(1-\beta\right)\Omega(a',a)f(a')\end{aligned}$$

$$\Omega(a',a) = \frac{c^{SP}(a')^{1-\sigma}}{\int c^{SP}(a'')^{1-\sigma} m(a'',a)da''}$$

2. FOC for the innovation rate:

$$\lambda(a) = \left\{ \frac{\overline{\phi}}{\gamma w_H^{SP}} V^{SP}(a) \right\}^{\frac{1}{\gamma-1}} \tag{30}$$

3. The distribution of products $f(a)$:

$$0 = -\frac{\partial}{\partial a} f(a) + (\lambda(a) - \delta_F) f(a) + \lambda_E \boldsymbol{\delta}(a) \tag{31}$$

4. The distributions of matched products $\tilde{m}(a'; a)$:

$$\begin{aligned} \frac{\partial}{\partial a} m(a'; a) = &- \frac{\partial}{\partial a'} m(a'; a) - (\delta_F + \delta_M) m(a'; a) + \frac{\zeta}{N_f} f(a') \\ &+ \zeta_0 \frac{\lambda_E}{N_f} \boldsymbol{\delta}(a') + \lambda(a') m(a'; a) \end{aligned} \tag{32}$$

   subject to the boundary condition $m(a'; 0) = \frac{\zeta_0}{N_f} f(a')$.

5. The cost shifter $c(a)$ satisfies

$$c^{SP}(a) = \left(\int \left(c^{SP}(a') \right)^{1-\sigma} m(a'; a) da' \right)^{\frac{1-\beta}{1-\sigma}} \tag{33}$$

6. The demand shifter for an age a product $D(a)$ satisfies

$$D^{SP}(a) = (1 - \beta) \int c(a')^{SP \frac{\beta}{1-\beta}(\sigma-1)} D^{SP}(a') m(a'; a) da' + 1 \tag{34}$$

7. The skilled wage w_H^{SP} satisfies the skilled labor market clearing condition

$$1 = \int \phi(\lambda(a)) f(a) da \tag{35}$$

C Proofs and derivation

Proof of Lemma 1

Given the production technology of intermediate good production, firm ω has the following marginal cost:

$$c(\omega) = w^{\beta}\left(\int_{\omega'\in\mathcal{S}(\omega)} p(\omega',\omega)^{1-\sigma}d\omega'\right)^{\frac{1-\beta}{1-\sigma}}$$

Inserting $p(\omega',\omega) = \mu c(\omega')$ and rewrite integral using the matched product distribution, we get

$$c\,(a) = w^{\beta}\left(\int \left(\mu c(a')\right)^{1-\sigma} dM(a';a)\right)^{\frac{1-\beta}{1-\sigma}}$$

Using Shephard's Lemma, Hicksian demand for ω from ω' can be obtained by

$$\begin{aligned}
x(\omega',\omega) &= \frac{\partial c(\omega')x(\omega')}{\partial p\,(\omega)}\\
&= \frac{\partial \log c(\omega')x(\omega')}{\partial p\,(\omega)} mc(\omega')x(\omega')\\
&= \frac{\partial}{\partial p\,(\omega)}\left[\log\left\{w^{\beta}\left(\int_{\omega''\in\mathcal{S}(\omega')} p(\omega'')^{1-\sigma}d\omega''\right)^{\frac{1-\beta}{1-\sigma}} x(\omega')\right\}\right]c(\omega')x(\omega')\\
&= \frac{1-\beta}{1-\sigma}\frac{\partial}{\partial p\,(\omega)}\left[\log\left(\int_{\omega''\in\mathcal{S}(\omega')} p(\omega'')^{1-\sigma}d\omega''\right)\right]c(\omega')x(\omega')\\
&= \frac{1-\beta}{1-\sigma}\frac{(1-\sigma)\,p(\omega)^{-\sigma}}{\int_{\omega''\in\mathcal{S}(\omega)} p(\omega'')^{1-\sigma}d\omega''}c(\omega')x(\omega')\\
&= (1-\beta)\,\frac{p(\omega)^{-\sigma}}{\int_{\omega''\in\mathcal{S}(\omega')} p(\omega'')^{1-\sigma}d\omega''}c(\omega')x(\omega') \qquad (36)
\end{aligned}$$

Note that

$$\begin{aligned}
c(\omega) &= w^{\beta}\left(\int_{\omega'\in\mathcal{S}(\omega)} p(\omega')^{1-\sigma}d\omega'\right)^{\frac{1-\beta}{1-\sigma}}\\
c(\omega)^{\frac{1-\sigma}{1-\beta}} &= w^{\frac{\beta}{1-\beta}(1-\sigma)}\int_{\omega'\in\mathcal{S}(\omega)} p(\omega')^{1-\sigma}d\omega'
\end{aligned}$$

$$\int_{\omega' \in \mathcal{S}(\omega)} p(\omega')^{1-\sigma} d\omega' = w^{\frac{\beta}{1-\beta}(\sigma-1)} c(\omega)^{\frac{1-\sigma}{1-\beta}}$$

Inserting this into (36),

$$\begin{aligned}
x(\omega', \omega) &= (1-\beta)\, p(\omega)^{-\sigma} c(\omega') x(\omega') w^{-\frac{\beta}{1-\beta}(\sigma-1)} c(\omega')^{\frac{\sigma-1}{1-\beta}} \\
x(\omega', \omega) &= (1-\beta)\, p(\omega)^{-\sigma} w^{-\frac{\beta}{1-\beta}(\sigma-1)} c(\omega')^{\frac{\sigma-\beta}{1-\beta}} x(\omega')
\end{aligned}$$

From the good market clearing condition,

$$\begin{aligned}
x(\omega) &= \int_{\omega' \in \mathcal{B}(\omega)} x(\omega', \omega) d\omega' + y(\omega) \\
&= \int_{\omega' \in \mathcal{B}(\omega)} (1-\beta)\, p(\omega)^{-\sigma} w^{-\frac{\beta}{1-\beta}(\sigma-1)} c(\omega')^{\frac{\sigma-\beta}{1-\beta}} x(\omega') d\omega' + y(\omega) \\
&= (1-\beta)\, p(\omega)^{-\sigma} w^{-\frac{\beta}{1-\beta}(\sigma-1)} \int_{\omega' \in \mathcal{B}(\omega)} c(\omega')^{\frac{\sigma-\beta}{1-\beta}} x(\omega') d\omega' + Y p(\omega)^{-\sigma} P^{\sigma}
\end{aligned}$$

Using the matched product distribution function

$$x(a) = (1-\beta)\, p(a)^{-\sigma} w^{-\frac{\beta}{1-\beta}(\sigma-1)} \int c(a')^{\frac{\sigma-\beta}{1-\beta}} x(a') dM(a'; a) + Y p(a)^{-\sigma} P^{\sigma}$$

Denote $D(a) = x(a) P^{1-\sigma} p(a)^{\sigma}$ as demand shifter for a,

$$\begin{aligned}
D(a) &= (1-\beta)\, w^{-\frac{\beta}{1-\beta}(\sigma-1)} \int c(a')^{\frac{\sigma-\beta}{1-\beta}} p(a')^{-\sigma} D(a') dM(a'; a) + PY \\
&= (1-\beta)\, \mu^{-\sigma} w^{-\frac{\beta}{1-\beta}(\sigma-1)} \int c(a')^{\frac{\beta}{1-\beta}(\sigma-1)} D(a') dM(a'; a) + PY
\end{aligned}$$

Finally, the ideal price index P is given by

$$\begin{aligned}
P &= \left(\int_{\omega \in \Omega} p^F(\omega)^{1-\sigma} d\omega \right)^{\frac{1}{1-\sigma}} \\
P &= \left(\int (\mu c(a))^{1-\sigma} dF(a) \right)^{\frac{1}{1-\sigma}}
\end{aligned}$$

Labor Demand and Labor Market Clearing for Production Worker

Using Shephard's Lemma, labor demand from ω can be obtained by

$$\begin{aligned} l(\omega) &= \frac{\partial c(\omega)}{\partial w} x(\omega) \\ &= \frac{\partial \log c(\omega) x(\omega)}{\partial w} c(\omega) x(\omega) \\ &= \frac{\partial}{\partial w} \left[\log \left\{ w^{\beta} \left(\int_{\omega' \in \mathcal{S}(\omega)} p(\omega')^{1-\sigma} d\omega' \right)^{\frac{1-\beta}{1-\sigma}} x(\omega) \right\} \right] c(\omega) x(\omega) \\ &= \beta \frac{\partial}{\partial w} \left[\log w \right] c(\omega) x(\omega) \\ &= \frac{\beta c(\omega) x(\omega)}{w} \end{aligned}$$

With a notation,

$$\begin{aligned} wl(a) &= \beta c(a) x(a) \\ &= \beta c(a) D(a) P^{\sigma-1} p(a)^{-\sigma} \\ &= \beta \mu^{-\sigma} \left(\frac{c(a)}{P} \right)^{1-\sigma} D(a) \end{aligned}$$

From unskilled labor market clearing condition,

$$w = \beta \mu^{-\sigma} \int \left(\frac{c(a)}{P} \right)^{1-\sigma} D(a) dF(a)$$

Decomposition of Value Function

Conjecture that the value function takes an additive form

$$V^F(n, a) = nV(a).$$

Then, 10 can be expressed by

$$rnV(a) - nV_t(a) = n\pi(a) - \delta_F nV(a) + nV_a(a) + \max_{\lambda \geq 0} \left[n\lambda V(a) - nw_H \phi(\lambda) \right]$$

Dividing both side by n,

$$r(t)V(a) - V_t(a) = \pi(a) - \delta_F V(a) + V_a(a) + \max_{\lambda \geq 0} \left[\lambda V(a,t) - w_H \phi(\lambda)\right]$$

Proof of Lemma 2.

Proof. Let define $\mu_p(a) = Y^{\frac{1}{\sigma}} c^{SP}(a)$ and $\mu_l = Y^{\frac{1}{\sigma}}$ as Lagrangian multipliers for (16) and (18). Then, first order conditions are

$$y(a) = c^{SP}(a)^{-\sigma}, \tag{37}$$

$$l(a) = \beta c(a)^{SP} x(a), \tag{38}$$

and

$$x(a', a) = (1-\beta)\left[c(a)^{SP}\right]^{-\sigma}\left[c(a')^{SP}\right]^{\frac{\sigma-\beta}{1-\beta}} x(a') \tag{39}$$

By substituting (38) and (39) into (16), we have

$$x(a) = \frac{1}{\beta^\beta (1-\beta)^{1-\beta}} \left(c(a)^{SP} x(a) \beta\right)^\beta \left(\int ((1-\beta)\left[c(a')^{SP}\right]^{-\sigma}\left[c(a)^{SP}\right]^{\frac{\sigma-\beta}{1-\beta}} x(a))^{\frac{\sigma-1}{\sigma}} m(a';a) da'\right)^{\frac{\sigma}{\sigma-1}(1-\beta)},$$

$$x(a) = x(a)\left[c(a)^{SP}\right]^\sigma \left(\int (c(a')^{-\sigma})^{\frac{\sigma-1}{\sigma}} m(a';a) da'\right)^{\frac{\sigma}{\sigma-1}(1-\beta)},$$

$$c(a) = \left(\int (c(a'))^{1-\sigma} m(a';a) da'\right)^{\frac{1}{1-\sigma}(1-\beta)}.$$

Next, by substituting (38) and (39) into (17), and with similar step in the proof of Lemma 1, we could derive:

$$D^{SP}(a) = (1-\beta)\int \left[c(a')^{SP}\right]^{\frac{\beta}{1-\beta}(\sigma-1)} D^{SP}(a') m(a';a) da' + 1,$$

where $D(a) = x(a)\left[P^{SP}\right]^{1-\sigma}\left[c(a)^{SP}\right]^\sigma$. Finally, from (18) we have:

$$1 = \beta \int \left(\frac{c(a)^{SP}}{P^{SP}}\right)^{1-\sigma} D^{SP}(a) f(a) da$$

■

D Social Planner's Problem

Objective function

From Lemma 2, using 21

$$
\begin{aligned}
&\text{argmax} \int_0^\infty \exp(-\rho t) \log\left(\int y(a)^{\frac{\sigma-1}{\sigma}} dF(a) \right)^{\frac{\sigma-1}{\sigma}} dt \\
&\text{argmax} \int_0^\infty \exp(-\rho t) \frac{\sigma-1}{\sigma} \log \underbrace{\left(\int c^{SP}(a)^{1-\sigma} dF(a) \right)}_{P^{1-\sigma}} dt \\
=&\text{argmax} \int_0^\infty \exp(-\rho t) \log\left(\beta \int c^{SP}(a)^{1-\sigma} D^{SP}(a) f(a) da \right) dt \\
=&\text{argmax} \int_0^\infty \exp(-\rho t) \log\left(\int c^{SP}(a)^{1-\sigma} D^{SP}(a) f(a) da \right) dt
\end{aligned}
$$

Let

$$
\begin{aligned}
\mathbf{x}(t) &= \left\{ f(a,t), m(a',a,t), c(a',t)^{1-\sigma}, D(a,t), \lambda(a,t) \right\}_{a,a' \geq 0} \\
\boldsymbol{\mu}(t) &= \left\{ \mu_f(a,t), \mu_m(a',a,t), \mu_{mf}(a',t), \mu_c(a,t), \mu_D(a,t), \mu_\lambda \right\}_{a,a' \geq 0}
\end{aligned}
$$

where is $\mathbf{x}(t)$ a set of control variables and $\boldsymbol{\mu}(t)$ is a set of shadow values.

The Hamiltonian is

$$
\begin{aligned}
\mathcal{H}\left(t, \mathbf{x}(t), \mathbf{y}(t), \boldsymbol{\mu}(t)\right) = \log&\left(\int c^{SP}(a,t)^{1-\sigma} D^{SP}(a,t) f(a,t) da \right) \\
&+ \int \mu_f(a,t) \left[-\frac{\partial}{\partial a} f(a,t) + (\lambda(a,t) - \delta_F) f(a,t) + \lambda_E \boldsymbol{\delta}(a) \right] da \\
&+ \int \int \mu_m(a',a,t) \left[-\frac{\partial}{\partial a} m(a',a,t) - \frac{\partial}{\partial a'} m(a',a,t) + (\lambda(a',t) - \delta_F - \delta_M) m(a',a,t) + \zeta \frac{f(a',t)}{N_f} + \frac{\zeta_0 \lambda_E}{N_f} \boldsymbol{\delta}(a') \right] d \\
&+ \int \mu_{mf}(a,t) \left[\zeta_0 \frac{f(a,t)}{N_f} - m(a,0,t) \right] da \\
&+ \int \mu_D(a,t) \left[(1-\beta) \int \left[c(a',t)^{SP} \right]^{\frac{\beta}{1-\beta}(\sigma-1)} D^{SP}(a') m(a',a,t) da' - D^{SP}(a,t) \right] da \\
&\cancel{+ \int \mu_c(a,t) \left[\left(\int c^{SP}(a',t)^{1-\sigma} m(a',a,t) da' \right)^{1-\beta} - c^{SP}(a,t)^{1-\sigma} \right] da}
\end{aligned}
$$

$$+ \mu_{\lambda}(t)\left[1 - \int \phi\left(\lambda(a,t)\right) f(a,t)da\right]$$

In the following, we focus on the stationary version of the social planner's problem. Note that

$$\begin{aligned}
\frac{c^{SP}(a)^{1-\sigma} D^{SP}(a)}{\int c^{SP}(a)^{1-\sigma} D^{SP}(a) f(a) da} &= \beta \frac{c^{SP}(a)^{1-\sigma} D^{SP}(a)}{\left(P^{SP}\right)^{1-\sigma}} \\
&= \beta \left(\frac{c^{SP}(a)}{P^{SP}}\right)^{1-\sigma} D^{SP}(a)
\end{aligned}$$

Next, we conjecture

$$\mu_D(a) = \left(\frac{c^{SP}(a)}{P}\right)^{1-\sigma} f(a)$$

To verify that, taking a first order condition w.r.t. $c(a)^{1-\sigma}$,

$$\begin{aligned}
0 =& \frac{D^{SP}(a) f(a)}{\int c^{SP}(a)^{1-\sigma} D^{SP}(a) f(a,t) da} - \beta \int \mu_D(a') \left[c^{SP}(a)\right]^{\frac{-1}{1-\beta}(\sigma-1)} D^{SP}(a) m(a;a') da' \\
=& \beta \frac{D^{SP}(a) f(a)}{P^{1-\sigma}} - \beta \int \frac{c^{SP}(a')^{1-\sigma} f(a')}{P^{1-\sigma}} \left[c^{SP}(a)\right]^{\frac{-1}{1-\beta}(\sigma-1)} D^{SP}(a) m(a;a') da'
\end{aligned}$$

So

$$\begin{aligned}
\beta D^{SP}(a) &= \beta \int c^{SP}(a')^{1-\sigma} \left[c^{SP}(a)\right]^{\frac{-1}{1-\beta}(\sigma-1)} D^{SP}(a) \frac{m(a;a') f(a')}{f(a)} da' \\
&= \beta \left[c^{SP}(a)\right]^{\frac{-1}{1-\beta}(\sigma-1)} D^{SP}(a) \int c^{SP}(a')^{1-\sigma} m(a';a) da' \\
&= \beta D^{SP}(a)
\end{aligned}$$

which is a desired result.

First order conditions w.r.t. $D^{SP}(a)$ gives

$$\mu_D(a) = \frac{c^{SP}(a)^{1-\sigma} f(a)}{\int c^{SP}(a)^{1-\sigma} D^{SP}(a) f(a) da} + (1-\beta) \int \mu_D(a') \left[c^{SP}(a)\right]^{\frac{\beta}{1-\beta}(\sigma-1)} m(a;a') da'.$$

So,

$$\left(\frac{c^{SP}(a)}{P}\right)^{1-\sigma} = \beta \left(\frac{c^{SP}(a)}{P}\right)^{1-\sigma} + (1-\beta) \int \left(\frac{c^{SP}(a')}{P}\right)^{1-\sigma} \left[c^{SP}(a)\right]^{\frac{\beta}{1-\beta}(\sigma-1)} \frac{f(a') m(a;a')}{f(a)} da'.$$

So,

$$c^{SP}(a)^{(1-\sigma)\frac{1}{1-\beta}} = \int c^{SP}(a')^{1-\sigma}\frac{f(a')m(a;a')}{f(a)}da'$$

So, we recovered cost functions:

$$c^{SP}(a)^{1-\sigma} = \left(\int c^{SP}(a')^{1-\sigma}m(a';a)da'\right)^{1-\beta} \tag{40}$$

First order conditions w.r.t. $f(a)$ gives

$$\begin{aligned}&\rho\mu_f(a) = \\ &\beta\left(\frac{c^{SP}(a)}{P}\right)^{1-\sigma}D^{SP}(a) + \frac{\partial}{\partial a}\mu_f(a) + \mu_f(a)\left(\lambda(a) - \delta_F\right) - \mu_\lambda\phi\left(\lambda(a)\right) + \frac{\zeta}{N_f}\int \mu_m(a,a')da'\end{aligned} \tag{41}$$

First order conditions w.r.t. $m(a',a)$ gives

$$\begin{aligned}\rho\mu_m(a',a) =& \frac{\partial}{\partial a}\mu_m(a',a) + \frac{\partial}{\partial a'}\mu_m(a',a) + \mu_m(a',a)\left(\lambda(a') - \delta_N - \delta\right) \\ &+ \mu_D(a)(1-\beta)\left[c^{SP}(a')\right]^{(\sigma-1)\frac{\beta}{1-\beta}}D^{SP}(a')\end{aligned}$$

So,

$$\begin{aligned}\rho\mu_m(a',a) =& \frac{\partial}{\partial a}\mu_m(a',a) + \frac{\partial}{\partial a'}\mu_m(a',a) + \mu_m(a',a)\left(\lambda(a') - \delta_N - \delta\right) \\ &+ (1-\beta)\left[c^{SP}(a')\right]^{(\sigma-1)\left(\frac{\beta}{1-\beta}\right)}D^{SP}(a')\left(\frac{c^{SP}(a)}{P}\right)^{1-\sigma}f(a)\end{aligned}$$

So,

$$\begin{aligned}\left(\rho + \delta_F + \delta_M - \lambda(a')\right)\mu_m(a',a) =& \frac{\partial}{\partial a}\mu_m(a',a) + \frac{\partial}{\partial a'}\mu_m(a',a) \\ &+ c^{SP}(a)^{1-\sigma}(1-\beta)\left[c^{SP}(a')\right]^{-(1-\sigma)\left(\frac{1}{1-\beta}\right)}f(a)\underbrace{\left(\frac{c^{SP}(a')}{P}\right)^{1-\sigma}D^{SP}(a')}_{\text{revenue of a}'}\end{aligned} \tag{42}$$

First order conditions w.r.t. $\lambda(a)$ gives

$$0 = \mu_f(a) f(a) + \int \mu_m(a, a') m(a, a') da' - \mu_\lambda \phi'(\lambda(a)) f(a)$$

So,

$$\lambda(a) = \phi'^{-1}\left(\frac{1}{\mu_\lambda}\left\{\mu_f(a) + \frac{\int \mu_m(a, a') m(a, a') da'}{f(a)}\right\}\right)$$

Note that

$$\phi'^{-1}(x) = \left(\frac{\overline{\phi}}{\gamma} x\right)^{\frac{1}{\gamma-1}}$$

Therefore,

$$\lambda(a) = \left[\frac{\overline{\phi}}{\gamma \mu_\lambda}\left\{\mu_f(a) + \frac{\int \mu_m(a, a') m(a, a') da'}{f(a)}\right\}\right]^{\frac{1}{\gamma-1}}$$

Define

$$\begin{aligned}
V^{SP}(a) &\equiv \mu_f(a) + \frac{\int \mu_m(a, a') m(a, a') da'}{f(a)} \\
V^{M}(a', a) &\equiv \mu_m(a', a) \\
w_H^{SP} &\equiv \mu_\lambda
\end{aligned}$$

From (42), (changing a and a')

$$\begin{aligned}
(\rho + \delta_F + \delta_M - \lambda(a)) \mu_m(a, a') = &\frac{\partial}{\partial a}\mu_m(a, a') + \frac{\partial}{\partial a'}\mu_m(a, a') \\
&+ c^{SP}(a')^{1-\sigma}(1-\beta)\left[c^{SP}(a)\right]^{-(1-\sigma)\left(\frac{1}{1-\beta}\right)} f(a')\left(\frac{c^{SP}(a)}{P}\right)^{1-\sigma} D^{SP}(a)
\end{aligned}$$

Multiply $m(a, a')$ and divide by $f(a)$ and integrate over a'

$$(\rho + \delta_F + \delta_M - \lambda(a)) \int \mu_m(a, a') \frac{m(a, a')}{f(a)} da' = \frac{\partial}{\partial a} \int \mu_m(a, a') \frac{m(a, a')}{f(a)} da'$$

$$+ (1-\beta)\, r^{SP}(a) \int \frac{c^{SP}(a')^{1-\sigma}}{\int c^{SP}(a'')^{1-\sigma} m(a'';a)da'} \frac{m(a,a')f(a')}{f(a)} da'$$

So

$$\begin{aligned}(\rho+\delta_F+\delta_M-\lambda(a)) \int \mu_m(a,a') \frac{m(a,a')}{f(a)} da' =& \frac{\partial}{\partial a} \int \mu_m(a,a') \frac{m(a,a')}{f(a)} da' \\ &+ (1-\beta)\, r^{SP}(a) \underbrace{\frac{\int c^{SP}(a')^{1-\sigma} m(a',a)da'}{\int c^{SP}(a'')^{1-\sigma} m(a'';a)da'}}_{=1}\end{aligned}$$

So,

$$\begin{aligned}(\rho+\delta_F+\delta_M-\lambda(a)) \int \mu_m(a,a') \frac{m(a,a')}{f(a)} da' =& \frac{\partial}{\partial a} \int \mu_m(a,a') \frac{m(a,a')}{f(a)} da' \\ &+ (1-\beta)\, r^{SP}(a),\end{aligned} \tag{43}$$

From (41) + (43)

$$(\rho+\delta_F-\lambda(a))\, V^{SP}(a) = r^{SP}(a) + \frac{\partial}{\partial a} V(a) - w_H^{SP} \phi(\lambda(a)) + \int V^M(a,a') \left\{ \frac{\zeta}{N_f} - \delta_M \frac{m(a,a')}{f(a)} \right\} da',$$

where

$$r^{SP}(a) = \left(\frac{c^{SP}(a)}{P^{SP}} \right)^{1-\sigma} D^{SP}(a).$$

From (42),

$$\begin{aligned}(\rho+\delta_F+\delta_M-\lambda(a))\, V^M(a,a') =& \frac{\partial}{\partial a} V^M(a,a') + \frac{\partial}{\partial a'} V^M(a,a') \\ &+ c^{SP}(a')^{1-\sigma} (1-\beta) \left[c^{SP}(a) \right]^{-(1-\sigma)\left(\frac{1}{1-\beta}\right)} f(a') r(a)\end{aligned}$$

$$\begin{aligned}(\rho+\delta_F+\delta_M-\lambda(a'))\, \mu_m(a',a) =& \frac{\partial}{\partial a} \mu_m(a',a) + \frac{\partial}{\partial a'} \mu_m(a',a) \\ &+ c^{SP}(a)^{1-\sigma} (1-\beta) \left[c^{SP}(a') \right]^{-(1-\sigma)\left(\frac{1}{1-\beta}\right)} f(a) \underbrace{\left(\frac{c^{SP}(a')}{P} \right)^{1-\sigma} D^{SP}(a')}_{\text{revenue of a'}}\end{aligned} \tag{44}$$

Inserting (40),

$$\left(\rho+\delta_F+\delta_M-\lambda\left(a\right)\right)V^M\left(a,a'\right)=\frac{\partial}{\partial a}V^M\left(a,a'\right)+\frac{\partial}{\partial a'}V^M\left(a,a'\right)+r^{SP}(a)\left(1-\beta\right)\Omega(a',a)f(a')$$

where

$$\Omega(a',a)=\frac{c^{SP}(a')^{1-\sigma}}{\int c^{SP}(a'')^{1-\sigma}m(a'',a)da''}$$

E Numerical Appendix

Product Distribution f

We solve the model with finite difference methods. Throughout this section, to construct the derivative matrices, we use a backward approximation when the drift of the state variable is positive, and a forward approximation when the drift of the state is negative. Notice that the stationary distribution is the solution for the following differential equation:

$$0=-\frac{\partial f\left(a\right)}{\partial a}+\left(\lambda(a)-\delta_N\right)f\left(a\right)+\lambda_E\boldsymbol{\delta}(a) \tag{45}$$

We now discretize a on an evenly spaced $N_a\times 1$. Let D_a be the $N_a\times N_a$ matrix that, when pre-multiplying f, gives an approximation of f_a. Analogously, define D_a:

$$f_a=D_af$$

Vectorize (45) and obtain,

$$f=-\left\{-D_a+\lambda-\delta_N\right\}^{-1}\lambda_E\boldsymbol{\delta}$$

where the element of N_a vector f consists of $f\left(a\right)$, the element of N_a vector λ consists of $\lambda\left(a\right)$, and the element of N_a vector $\boldsymbol{\delta}$ consists of $\boldsymbol{\delta}(a)$

Matched Product Distribution m

The distributions of matched products $m(a';a)$ is given by

$$\frac{\partial}{\partial a}m(a';a)=-\frac{\partial}{\partial a'}m(a';a)+\left(\lambda\left(a'\right)-\delta_N-\delta\right)m(a';a)+\frac{\zeta_0\lambda_E}{N}f(a')+\zeta_0\frac{\lambda_E}{N_f}\boldsymbol{\delta}(a')$$

subject to the boundary condition $m(a';0) = 0$. Vectorize this, and obtain

$$
\begin{aligned}
\frac{m - m_{-da}}{da} &= \left(-D_a + \lambda\left(a'\right) - \delta_N - \delta\right) m + \frac{\zeta_0 \lambda_E}{N} f + \zeta_0 \frac{\lambda_E}{N_f} \boldsymbol{\delta} \\
m - m_{-da} &= da \left(-D_a + \lambda\left(a'\right) - \delta_N - \delta\right) m + da \frac{\zeta_0 \lambda_E}{N} f + da \zeta_0 \frac{\lambda_E}{N_f} \boldsymbol{\delta} \\
m &= \left\{I_a - da\left(-D_a + \lambda\left(a'\right) - \delta_N - \delta\right)\right\}^{-1} \left(m_{-da} + da \frac{\zeta_0 \lambda_E}{N} f + da \zeta_0 \frac{\lambda_E}{N_f} \boldsymbol{\delta}\right)
\end{aligned}
$$

Starting from $m(\cdot;0) = 0$, the forward iteration of the above vectorized equation gives the distributions of matched products $m(a';a)$ for each a. Note that $m(a';a)$ converges as a becomes sufficiently large. Therefore, we only need to forward iterate until $m(a';a)$ converges.

Value Function V

Let Δ denote step-size and τ the iteration of the algorithm. Then given $V^{\tau-1}(a)$, the implicit method gives an update

$$
\frac{1}{\Delta}\left(V^{\tau}(a) - V^{\tau-1}(a)\right) + \left(\rho + \delta_N\right) V^{\tau}(a) = \pi(a) + V^{\tau}_a(a) + \lambda(a) V^{\tau}(a) - w_H \phi(\lambda(a))
$$

where $\pi(a) = \left(1 - \frac{1}{\mu}\right)\left(\frac{\mu c(a)}{P}\right)^{1-\sigma} D(a)$. Rearranging this,

$$
\left(\frac{1}{\Delta} + \rho + \delta_N - \lambda(a)\right) V^{\tau}(a) - V^{\tau}_a(a) = \pi(a) - w_H \phi(\lambda(a)) + \frac{1}{\Delta} V^{\tau-1}(a)
$$

Now, we vectorize the HJB equation:

$$
\begin{aligned}
\left(\frac{1}{\Delta} + \rho + \delta_N - \lambda - D_a\right) V^{\tau} &= \pi + \frac{1}{\Delta} V^{\tau-1} \\
V^{\tau} &= \left\{\frac{1}{\Delta} + \rho + \delta_N - \lambda - D_a\right\}^{-1} \left(\pi + \frac{1}{\Delta} V^{\tau-1}\right)
\end{aligned}
$$

where the element of N_a vector V^{τ} consists of $V^{\tau}(a)$ and the element of N_a vector π consists of $\pi(a)$. The implicit method works by updating V^{τ} through the above equation.

Social Matching Value Function V^M

$$
\left(\rho + \delta_F + \delta_M - \lambda(a)\right) V^M(a, a') = \frac{\partial}{\partial a} V^M(a, a') + \frac{\partial}{\partial a'} V^M(a, a') + \Omega(a', a)\left(1 - \beta\right) r^{SP}(a)
$$

We now discretize a' on an evenly spaced $N_a \times 1$ grid and a on an evenly spaced $N_a \times 1$. Stack these according to:

$$\begin{pmatrix} a_1, a'_1 \\ a_2, a'_1 \\ \vdots \\ a_{N_a}, a'_1 \\ \vdots \\ a_1, a'_{N_a} \\ \vdots \\ a_{N_a}, a'_{N_a} \end{pmatrix}$$

$$\begin{aligned} \left(\rho + \delta_F + \delta_M - \lambda\left(a\right) - D_a - D_{a'}\right) V^M &= A \\ V^M &= \left(\rho + \delta_F + \delta_M - \lambda\left(a\right) - D_a - D_{a'}\right)^{-1} A \end{aligned}$$

where the element of $N_a \times N_a$ vector V_m consists of $V_m\left(a, a'\right)$, and the element of $N_a \times N_a$ matrix A consists of

$$A = \Omega(a', a)\left(1 - \beta\right) r^{SP}(a)$$