Production Networks and R&D Allocation

Yasutaka Koike-Mori
UNC Chapel Hill

Koki Okumura
UCLA*

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Abstract

This paper investigates how production networks shape firms' R&D decisions and the resulting aggregate inefficiencies. We develop a dynamic model in which firms form supplier relationships through an exogenous yet persistent matching process and rely on those links to introduce new products. The model features two sources of misallocation: market-power distortions and a network-formation externality whereby firms fail to internalize that their R&D makes them more attractive trading partners, improving match quality for all firms in the economy. We estimate the model using Japanese firm-to-firm transaction and patent data and show that the first-best allocation lies close to the decentralized outcome. Market-power corrections raise R&D incentives for older firms by relieving double marginalization along long supply chains. Internalizing the network-formation externality instead tilts R&D toward younger firms that expand their supplier base most rapidly. These opposing forces offset each other, leaving the planner's allocation near the observed one.

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1 Introduction

Firms' research and development (R&D) decisions are fundamentally shaped by their position within production networks. When Toyota develops new automotive models, for instance, it does not innovate in isolation but leverages an established web of engine and chassis suppliers. This interconnectedness means that a firm's R&D choices are intrinsically linked to its trading partners.

While the literature has extensively studied how innovation spreads through knowledge networks (Liu and Ma, 2023), the channel through which the *production network itself* shapes R&D incentives remains less explored. Understanding this channel and the inefficiencies that may arise is crucial for optimal innovation policy. This paper investigates these dynamics by developing a model that captures the interplay between production networks and R&D decisions and then quantifies the resulting welfare effects.

Motivated by empirical patterns from a unique Japanese dataset, we build a dynamic model with three core features. First, firms build networks of suppliers over their lifecycle through an exogenous matching process that combines random matching with relationship persistence. This process generates an age-dependent network structure where older firms accumulate larger and more valuable supplier networks. Second, firms invest in R&D to create new product varieties. Third, and most critically, successful innovation leverages existing network connections: new products are produced using established suppliers and sold through existing customer relationships.

This framework reveals two primary sources of R&D misallocation. The first stems from standard market power distortions: markups and double marginalization reduce private returns to innovation, particularly for older firms embedded in long supply chains. The second represents our novel contribution: a *network-formation externality*. When firms innovate, they become more productive and desirable trading partners, improving the expected quality of potential matches for all other firms in the market. However, innovating firms are not compensated for this contribution and fail to internalize the surplus that accrues to their suppliers when relationships form.

Our quantitative analysis shows that the social planner's optimal R&D allocation is nearly identical to the decentralized equilibrium allocation. This occurs because two forces create counteracting effects on older firms' R&D incentives. The first force arises from correcting market power distortions. Eliminating markups and double marginalization boosts the relative value of older firms in long supply chains, pushing R&D resources *toward* them. The second force comes from internalizing the network-formation externality. Since this externality is most significant for younger firms rapidly building their networks, internalizing it pushes R&D resources *away* from older firms. These forces almost perfectly offset each other.

Related Literature

This paper connects the literature on the macroeconomic importance of networks, as typified by Acemoglu and Carvalho (2012), to the literature on R&D as the creation of new goods, for example, Romer (1990) and Klette and Kortum (2004). Our primary contribution is to integrate these two strands, showing how the transactional structure of a production network shapes R&D incentives and creates a novel set of allocative inefficiencies.

The production networks literature has demonstrated that network structure serves as a potent amplifier of economic shocks while creating misallocation in factor markets. Research by Liu (2019), Baqaee and Farhi (2020), Bigio and La'O (2020), Baqaee et al. (2023), and Osotimehin and Popov (2023) has shown how market power and markup heterogeneity, transmitted through input-output linkages, generate substantial aggregate welfare losses. While our analysis of market power distortions builds directly on these insights, we push the boundaries of this mechanism beyond static labor allocation to encompass the allocation of R&D resources.

A growing body of work examines the endogenous evolution of these networks, focusing on link formation through partner search and matching to explain the economy's structure (see Oberfield (2018), Arkolakis et al. (2023), Eaton et al. (2022), Huneeus (2018), Bernard et al. (2022), Acemoglu and Azar (2020)). Our model shares this dynamic perspective, with network connections evolving as firms mature, a pattern also explored by Aekka and Gaurav (2023) and Asai and January (2025). Where we differ is in shifting the central margin from partner search to R&D investment, thereby identifying a previously unrecognized externality: the network-formation externality.

This externality provides a new perspective on R&D misallocation, distinct from inefficiencies arising through "knowledge networks" studied by Liu and Ma (2023), Cai and Li (2019), and Cai and Tian (2021). Our mechanism operates through the production network: the inefficiency stems not from imitation but from innovators' inability to capture the value they create by becoming better trading partners. This differs from other R&D misallocation sources, including crowding out (Acemoglu et al. (2018)) or innovation spillovers (Aghion et al. (2023), Ayerst (2023)). The interaction between our externality and market power distortions produces our main result: two opposing forces that largely offset each other. [1].

[1] Empirical work using inter-firm network data (Boehm et al. (2019), Bernard et al. (2022), Bernard et al. (2019), Carvalho et al. (2020), Bai et al. (2023), Daisuke (2017)) has focused on short-run shock propagation.

The rest of the paper is organized as follows. Section 2 presents the data and empirical findings on the relationship between production networks, firm age, and R&D. Section 3 develops a dynamic model of production networks and R&D based on these empirical findings. Section 4 analyzes the misallocation of R&D workers in the decentralized equilibrium compared to the social planner's solution. Section 5 presents a quantitative exercise, estimating the model parameters and comparing the model's predictions with the data. Finally, Section 6 concludes the paper.

2 Motivating Facts on Production Networks and Innovation

Using a dataset of B2B transactions and patents as a proxy for R&D, we document facts that govern the dynamics of supply chains. Older firms use intermediate goods more intensively and maintain wider networks with more buyers and suppliers. These firms are also more interconnected with other older firms. Furthermore, we find that an exogenous increase in the R&D of a trading partner, as measured by patents, promotes R&D in the focal firm. We build a dynamic model in Section 3 based on these empirical regularities.

2.1 Data Sources

Our analysis relies on three main datasets: the Teikoku Databank (TDB) from Japan, the IIP Patent Database, and the OECD Triadic Patent Families (TPF) database.

First, the Teikoku Databank is a private credit research company that gathers information while preparing credit research reports on potential suppliers and buyers. This information includes a series of corporate-level characteristics along with the identities of the companies' suppliers and buyers. Notably, this database is not limited to publicly listed companies, offering broader coverage than databases like Compustat.

The second dataset is the IIP Patent Database, developed for patent statistical analysis using standardized data from the Patent Office. As the IIP Patent Database lacks corporate information, it is linked with the NISTEP database, which connects patent data to company names and identification numbers. Identification numbers from NISTEP then allow us to merge the IIP Patent Database with the TDB. For firms without identification numbers, we integrate the databases as much as possible using addresses and names.

The third dataset is the OECD Triadic Patent Families (TPF) database, which combines patent applications filed with the European Patent Office (EPO), the Japan Patent Office

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Our dataset, combining firm-level linkages with patent records, provides the first evidence on long-term supply chain and innovation dynamics.

(JPO), and the United States Patent and Trademark Office (USPTO) into patent families based on common priority applications. The primary data source for the TPF is the EPO's Worldwide Patent Statistics Database, which provides harmonized and comparable information on patents from the EPO, JPO, and USPTO. We use information on the IPC (four digits), the year of application (earliest filing date at the Japan Patent Office), and the nationality of the applicant for each patent.

2.2 Relationship between supply chain and firm age

We explore the empirical regularities that govern the time evolution and cross-sectional distribution of production networks to model the inter-firm matching process. The main finding is that the number of suppliers and buyers increases with firm age, but the rate of increase diminishes as firms grow older. In the cross-section, older firms are more likely to have older trading partners as well, mainly because trading relationships are sticky.

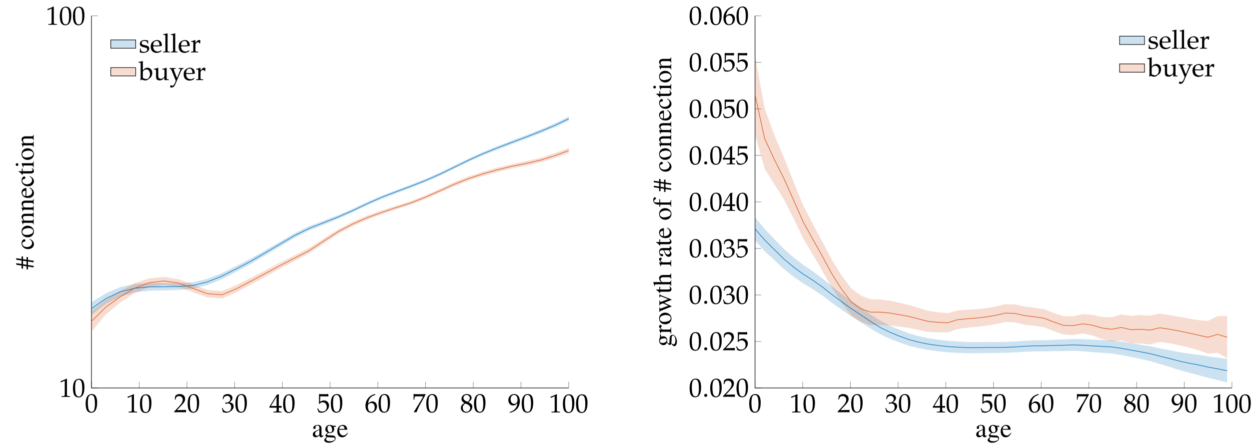
2.2.1 Age-dependent network relationships

We begin by analyzing the cross-sectional relationship between firm age and the number of trading partners. Figure 1a shows local linear regressions of the numbers of suppliers and buyers on firm age. Both series increase monotonically with age. After a steep increase in the first half of the life cycle, the pattern transitions to a more gradual, roughly log-linear rise. Figure 1b plots the corresponding growth rates, confirming that partner growth is high early in the life cycle and converges to a constant value. The number of suppliers tends to slightly exceed the number of buyers, but the slopes are not markedly different.

2.2.2 Positive age assortative matching by age in cross-section

Next, we turn to the features of trading partners conditional on age. Figure 2a plots trading partners' ages against firms' own ages. Older firms tend to match with older partners, indicating positive assortative matching by age in the cross-section.

To understand this assortative matching, we consider existing links (stocks) and the formation or withdrawal of business relationships (flows). We first examine the age heterogeneity in current-period matches within the flows. Figure 2b plots a local linear regression of partners' ages on firms' own ages. The flows exhibit only one to two years of heterogeneity, considerably smaller than the heterogeneity in the cross-section. Next, to examine the age dependence of link termination, Figure 3 plots the survival of each link as a function of the number of trading years between firms for each supplier and buyer age



(a) Number of connections by age

(b) Growth rate of connections by age

Figure 1: Lifecycle of production networks.

Notes: The figure shows local linear regression estimates relating firms' connections to firm age with controls. The sample covers 1998–2019. The control terms include prefecture, industry, and year fixed effects. The two lines represent the supplier and buyer sides as separate regressions. Shaded areas indicate the 95 percent confidence intervals.

quartile group. The probability that a link breaks decreases with the number of transaction years, but there is little significant heterogeneity across seller and buyer age groups.

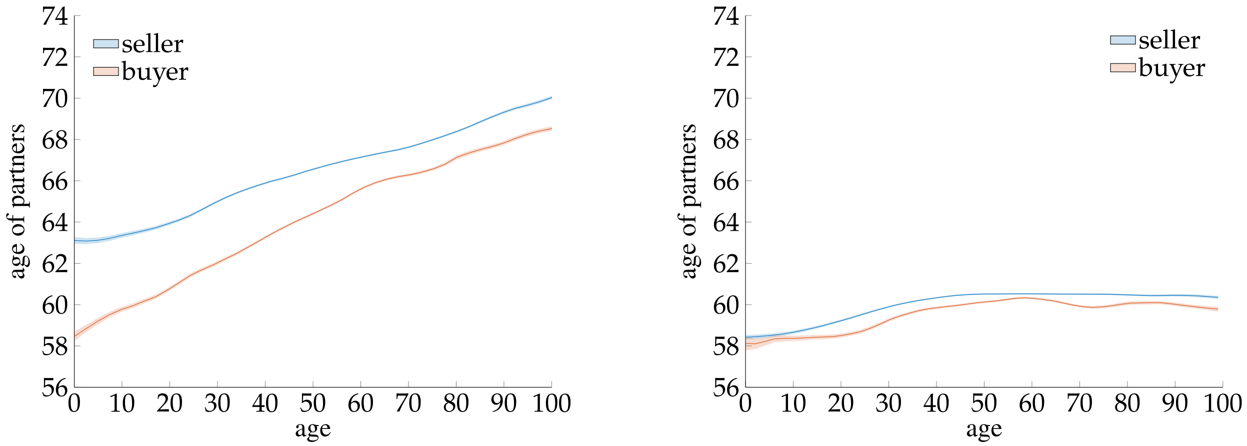
Why does age assortativity arise, even though the formation of new links is independent of firm age? The reason is that when business relationships are sticky, firms and their past counterparts age together. According to Figure 3, it takes more than 10 years for 90% of business relationships to dissolve. This high degree of stickiness generates the observed positive age assortativity of matching.

Finally, Figure 4 presents local linear regressions relating R&D activity to firm age. Both proxy measures of R&D activity increase monotonically with age.

2.3 Firm-Level Production Networks and R&D

In this section, we use an integrated database that combines firm-level transaction data from TDB with patent information from the IIP Patent Database maintained by the JPO to examine how the R&D activity of network-connected firms affects a focal firm's own innovation outcomes.

While our formulation builds on Acemoglu et al. (2015) and Liu and Ma (2023), it differs in two key respects. First, instead of aggregated industry- or technology-class variables, we exploit firm-level transaction links alongside patent citation relationships. Second, whereas Liu and Ma (2023) document technological spillovers and Acemoglu et al. (2015)



(a) Partners' age by firm age

(b) Partners' age in flows by firm age

Figure 2: Positive age assortativity.

Notes: The figure shows local linear regression estimates relating trading partners' ages to firm age with controls. The sample covers 1998–2019. The control terms include prefecture, industry, and year fixed effects. The two lines represent the supplier and buyer sides as separate regressions. Shaded areas indicate the 95 percent confidence intervals.

study foreign patent shocks transmitted through production networks, our data allow us to separate transactional and technological ties at the firm level. Appendix Section A.1 documents that the two networks still overlap, even when observed at the firm level.

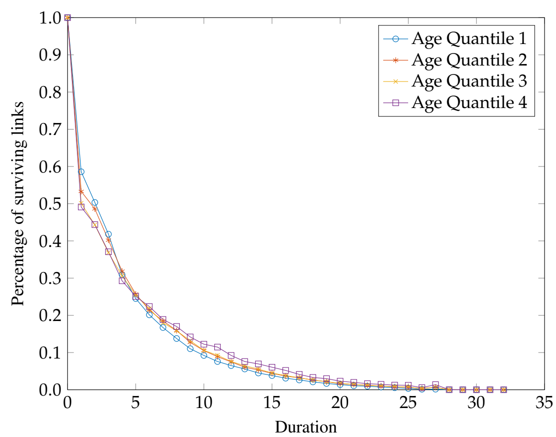
We estimate regressions on firm-level data from 1994 to 2019, measuring each firm's R&D activity by the number of patent applications and identifying production-network connections through the TDB database. Our baseline specification is

$$\log |\text{Patents}|_{i,t} = \beta_1 \log \frac{|\text{Sellers' Patents}|_{i,t}}{|\text{Sellers}|_{i,t}} + \beta_2 \log \frac{|\text{Buyers' Patents}|_{i,t}}{|\text{Buyers}|_{i,t}} + \text{controls}_{i,t} + \varepsilon_{i,t},$$

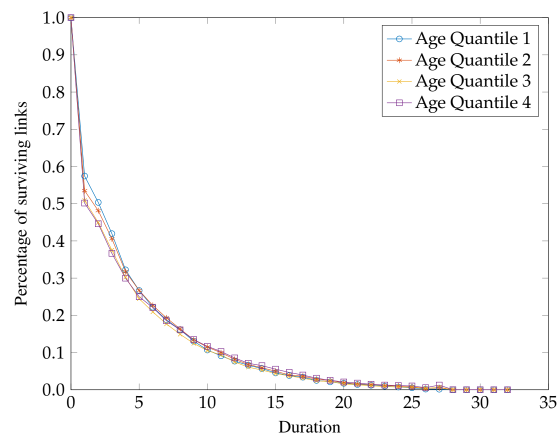
where the controls include firm and year fixed effects. Here, $|\text{Sellers' Patents}|_{i,t}$ ($|\text{Buyers' Patents}|_{i,t}$) denotes the total number of patent applications filed in year t by all sellers (buyers) of firm i. The ratios $|\text{Sellers' Patents}|_{i,t} / |\text{Sellers}|_{i,t}$ and $|\text{Buyers' Patents}|_{i,t} / |\text{Buyers}|_{i,t}$ therefore capture the average R&D activity of a firm's trading partners. To mitigate endogeneity concerns, we construct instrumental variables based on exposure to foreign patents:

$$\text{IV for } \frac{|\text{Sellers' Patents}|_{i,t}}{|\text{Sellers}|_{i,t}} = \sum_c \frac{|\text{Sellers' Patents}|_{i,c,t-1}}{|\text{Sellers' Patents}|_{i,t-1}} \times |\text{Foreign Patents}|_{c,t-1},$$

$$\text{IV for } \frac{|\text{Buyers' Patents}|_{i,t}}{|\text{Buyers}|_{i,t}} = \sum_c \frac{|\text{Buyers' Patents}|_{i,c,t-1}}{|\text{Buyers' Patents}|_{i,t-1}} \times |\text{Foreign Patents}|_{c,t-1}.$$



(a) Survival links by seller's age



(b) Survival links by buyer's age

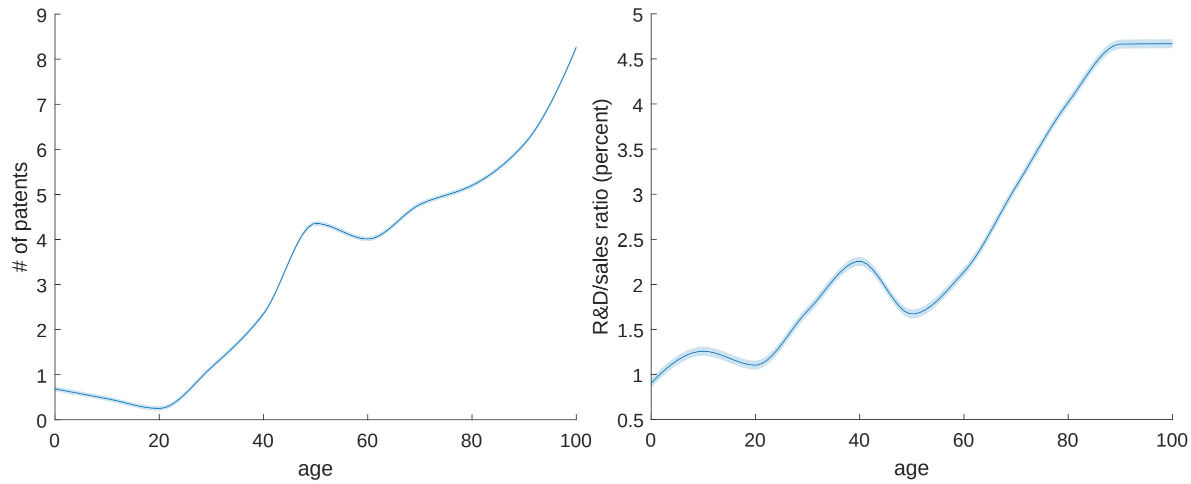
Figure 3: Survival links.

Notes: This figure plots the duration of business relationships between firms. To mitigate bias from sample observability, the sample is restricted to connections where (1) the connection is first observed more than one year after the period in which both firms were first observed and (2) the connection is no longer observed more than one year before the period in which either firm was last observed. The figure on the left classifies business relationships by the age quartiles of supplier firms. The right figure classifies business relationships by the age quartiles of buyer firms.

Here, $\left|\text{Foreign Patents}\right|_{c,t-1}$ is the number of triadic patents in class c filed at time $t-1$ at the Japan Patent Office by foreign firms, after excluding Japanese applicants. Triadic patents capture global technology trends rather than domestic demand shocks, yielding exogenous shifters for Japanese firms. By observing firm-level technology exposure directly, we can construct the above instruments without sector-level aggregation. The weights $|\text{Sellers' Patents}|_{i,c,t-1} / |\text{Sellers' Patents}|_{i,t-1}$ and $\left|\text{Buyers' Patents}\right|_{i,c,t-1} / \left|\text{Buyers' Patents}\right|_{i,t-1}$ capture the exposure of each firm to patent class c through its sellers and buyers, respectively.

Table 1 reports ordinary least squares (OLS) and instrumental variable (IV) estimates. In both sets of regressions the coefficients on the average number of sellers' and buyers' patent applications are positive and statistically significant, indicating that a firm's R&D activity increases with the innovation intensity of its trading partners. The IV estimates in columns (4)–(6) are larger in magnitude than the corresponding OLS estimates, consistent with downward bias in the naive specification.

To verify that the results stem from production-network mechanisms rather than direct technological spillovers, Table 2 shows that the coefficients remain sizable and significant even when we exclude patents that cite the applicant's buyers or sellers. This pattern supports the interpretation that production relationships, rather than shared patent classes, drive the observed spillovers.



(a) The number of patents

(b) R&D/Sales ratio

Figure 4: R&D and age.

Notes: The figure shows local linear regression estimates relating R&D measures to firm age with controls. The sample covers 1998–2019. The control terms include prefecture, industry, and year fixed effects for the firm and its clients. Shaded areas indicate the 95 percent confidence intervals.

Table 2: Regression Results (excluding patenting relationships)

	(1)	(2)	(3)	(4)	(5)	(6)
$\log \frac{\lvert\text{Sellers' Patents}\rvert_{i,t}}{\lvert\text{Sellers}\rvert_{i,t}}$	0.018***		0.017***	0.049**		0.043**
	(0.003)		(0.003)	(0.019)		(0.019)
$\log \frac{\lvert\text{Buyers. Patents}\rvert_{i,t}}{\lvert\text{Buyers}\rvert_{i,t}}$		0.020***	0.019***		0.081***	0.079***
		(0.003)	(0.003)		(0.023)	(0.023)
year fixed effect	Yes	Yes	Yes	Yes	Yes	Yes
firm fixed effect	Yes	Yes	Yes	Yes	Yes	Yes
Fstat				805.3	690.5	339.0
Observations	51,338	51,338	51,338	51,338	51,338	51,338

Standard errors in parentheses

* p<0.10, ** p<0.05, *** p<0.01

Notes: Standard errors are clustered at the industry × year level.

In sum, we provide novel evidence on the impact of production networks on firm innovation using a unique integrated database of firm-level transactions and patent data. Firms innovate more when their suppliers and buyers do so, even after addressing

Table 1: Regression Results

	(1)	(2)	(3)	(4)	(5)	(6)
$\log \frac{\lvert\text{Sellers' Patents}\rvert_{i,t}}{\lvert\text{Sellers}\rvert_{i,t}}$	0.020***		0.019***	0.068***		0.062***
	(0.003)		(0.003)	(0.018)		(0.018)
$\log \frac{\lvert\text{Buyers. Patents}\rvert_{i,t}}{\lvert\text{Buyers}\rvert_{i,t}}$		0.022***	0.020***		0.077***	0.073***
		(0.003)	(0.003)		(0.021)	(0.021)
year fixed effect	Yes	Yes	Yes	Yes	Yes	Yes
firm fixed effect	Yes	Yes	Yes	Yes	Yes	Yes
Fstat				840.4	759.8	373.5
Observations	53,553	53,553	53,553	53,553	53,553	53,553

Standard errors in parentheses

* p<0.10, ** p<0.05, *** p<0.01

Notes: Standard errors are clustered at the industry × year level.

endogeneity with instrumental variables. These findings indicate that production networks shape innovation incentives above and beyond technological knowledge spillovers.

3 A Model of Production Networks and R&D

Building on the empirical findings from Section 2, we develop a model that captures the interaction between production networks and innovation. Specifically, we build on Klette and Kortum (2004), a canonical model of innovation and firm dynamics, and introduce a firm-to-firm matching process replicating the age-dependent production network pattern observed in the data. Our original and most critical assumption is that successful R&D can use existing production networks to create and sell new goods, consistent with the positive impact of trading partners' R&D on a firm's own R&D found in the data.

3.1 Settings

A unit measure of infinitely lived households supplies a unit measure of production workers, R&D workers, and entrepreneurs[2]. Households have preferences over a final consumption good, $U_0 = \int_0^\infty \exp\left(-\rho t\right) \log Y\left(t\right) dt$, where $\rho > 0$ is the discount rate and $Y(t)$

[2]Fixing the supply of skilled labor abstracts from the underinvestment that would arise if other workers could perform R&D or if output were directly convertible into R&D. Recent work on R&D misallocation (Aghion et al., 2023; Liu and Ma, 2023) adopts the same assumption.

is consumption. The budget constraint is $\dot{A}(t) \leq r(t)A(t) + w(t) + w_H(t) + w_E(t) - P(t)Y(t)$ with the standard no-Ponzi condition, where $A(t)$ is the asset position, $r(t)$ is the interest rate, and $w(t)$, $w_H(t)$, and $w_E(t)$ denote wages for each type of worker. We set nominal GDP as the numeraire. The Euler equation then implies $r(t) = \rho$. In what follows, we focus on the stationary equilibrium and drop time subscripts when no confusion arises.

Products, Firms, and Production Networks

There is a continuum of intermediate goods, indexed by $\omega \in \Omega(t)$. The measure of intermediate goods evolves through the creation of new varieties and exit. Each intermediate good ω is produced by a monopolist, and a given firm may own multiple product lines and operate several goods simultaneously. Consider a firm $f \in \mathcal{F}$ that owns product line ω. Let $n(f)$ denote the number of product lines owned by firm f. We drop the firm subscript when no confusion arises. Denote by $\mathcal{S}(\omega) \subset \Omega$ the set of products used as inputs for good ω. Although buyers can be inferred from the inverse mapping of $\mathcal{S}(\cdot)$, for notational simplicity we write $\mathcal{B}(\cdot) : \Omega \to \Omega$ for the set of buyers in Ω.

Production Structure Given Networks

Firms use production workers and intermediate inputs for production. Intermediate inputs are imperfect substitutes with a constant elasticity of substitution, $\sigma \geq 1$. Production workers and the composite of intermediate inputs are combined in a Cobb-Douglas aggregator with labor share, β ($0 \leq \beta \leq 1$):

$$x(\omega) = \frac{1}{\beta^\beta \left(1-\beta\right)^{1-\beta}} l\left(\omega\right)^\beta \left(\int_{\omega' \in \mathcal{S}(\omega)} x(\omega', \omega)^{\frac{\sigma-1}{\sigma}} d\omega'\right)^{\frac{\sigma}{\sigma-1}(1-\beta)}, \tag{1}$$

where $l\left(\omega\right)$ is demand for production workers used to produce ω; $x(\omega', \omega)$ is the demand for product ω' used to produce ω.

A representative household has a CES utility function with an elasticity of substitution σ, which is the same elasticity as that for production:

$$Y = \left(\int_{\omega \in \Omega} y(\omega)^{\frac{\sigma-1}{\sigma}} d\omega\right)^{\frac{\sigma}{\sigma-1}}, \tag{2}$$

where $y(\omega) = Y\left(\frac{p(\omega)}{P}\right)^{-\sigma}$ is the final demand for product ω. For all ω, $x(\omega)$ satisfies the following market clearing condition:

$$x(\omega) = \int_{\omega' \in \mathcal{B}(\omega)} x(\omega, \omega') d\omega' + y(\omega), \tag{3}$$

Under monopolistic competition, suppliers charge a constant markup over their marginal cost:

$$p(\omega', \omega) = p^y(\omega') = \mu c(\omega'), \tag{4}$$

where $p(\omega', \omega)$ is the product price that supplier ω' charges to firm ω, $p^y(\omega')$ is the price that supplier ω' charges to the final good consumer, $c(\omega)$ is the marginal cost of product ω, and $\mu = \frac{\sigma}{\sigma - 1}$.

R&D, Matching, Entry, and Exit

Firms conduct R&D to acquire product lines of new variety[3]. We make the following assumptions about the set of production networks of sellers and buyers for which new products are available.

Assumption 1. *For a new product line ω' derived from the existing firm's product line ω,*

$$\mathcal{S}(\omega') = \mathcal{S}(\omega),$$

$$\mathcal{B}(\omega') = \mathcal{B}(\omega)$$

In words, existing firms can develop new products and sell them to existing buyers by using intermediate goods from suppliers of each product line owned by the existing product line.

The product level cost function follow Klette and Kortum (2004):

$$\tilde{\phi}(x, n) = n w_H \phi(\lambda), \tag{5}$$

where $\phi(\lambda) = \frac{1}{\phi} \lambda^\gamma$ with efficiency parameter $\phi > 0$, λ is a per product innovation rate, and $\gamma > 1$[4].

[3] Our model is based on Klette and Kortum (2004), but abstracts from productivity differences and growth. So instead of creative destruction, we assume variety creation.

[4] This product level cost function can be micro-founded by a constant-returns-to-scale innovation technology that uses R&D workers and the number of products as inputs. Namely, a firm f hires $l_H(f)$ units of skilled workers to add one more product at the flow rate $\Lambda(f) = n(f)^{1-1/\gamma} \left(\phi l_H(f)\right)^{1/\gamma}$, where Λ is the firm-level flow rate.

Firms are matched randomly with other firms at exogenous rates, ζ. The match is exogenously terminated at the rate δ_M or if one of the firms in either side of the match exits. Firms die at an exogenous rate δ_F. Entrepreneurs have access to a linear entry technology, where each R&D worker generates a flow of λ_E. Entrants start with ζ_0 mass of randomly chosen sellers or buyers.

3.2 Characterization of decentralized Equilibrium

The model involves the problem of tracking trading relationships between continuous products, which is impossible to solve in general, but we show that under our setting the equilibrium conditions are summarized by the following functions over the state space of ages, a.

Thereafter, let $F(a)$ be the cumulative density of products with respect to age a, and we will use a instead of ω. Let $N(t)$ be the total measure of products. Let $F(a,t)$ be defined so that the fraction of products which is owned by firms with age less than or equal to a at time t is $F(a,t)/N(t)$. Let $N_f(t)$ denote the total mass of firms. Because the total mass of firms evolves according to $\dot{N}_f(t) = \lambda_E - \delta_F N_f(t)$, in the steady state $N_f = \lambda_E/\delta_F$. We begin with a matching distribution. As we will show later, because the optimal innovation rate per product, λ, depends on age, the matching process satisfies the following differential equation.

Proposition 1. *Law of motion of matching process: The distribution of matched products with age less than a' connected with an age a product is given by*

$$\underbrace{\frac{\partial}{\partial a'}m(a',a) + \frac{\partial}{\partial a}m(a',a)}_{\text{time evolution of the matching distribution}}$$
$$= \underbrace{-(\delta_M + \delta_F)\,m(a',a)}_{\text{link destruction}} + \underbrace{\frac{\zeta}{N_f}f(a')}_{\text{random matching}} + \underbrace{\underbrace{\lambda(a')}_{\text{innovation rate}}\, m(a',a) + \frac{\zeta_0}{N_f}\lambda_E\boldsymbol{\delta}(a)}_{\text{new products created by partners of age } a'}, \tag{6}$$

where $\lambda(a)$ is the innovation rate of age a firm, subject to the boundary conditions

$$m(a';0) = \frac{\zeta_0}{N_f}f(a') \tag{7}$$

The matching distribution evolves over time because of the following reasons. First, the measure of matched products increases when the currently linked firm creates a new variety at rate $\lambda(a')$. Second, the matched products are lost when the link is terminated at the exogenous rate δ_M or when the linked firm exits at the exogenous rate δ_F. Finally, the matched products are added when there is random matching with new firms.

In general, if connections change stochastically, one needs to track changes in connections among countless firms each period. However, from the perspective of connected partners, the randomness disappears due to the law of large numbers. Thus, the distribution can be summarized by age. This formulation opens new possibilities for modeling two-sided production network structures.

Next, armed with a matching distribution as a function of age, we can characterize the equilibrium objects of firms using the demand shifters $D(a)$ and the cost shifters $c(a)$. These functions solve the two fixed-point problems described in the following Lemma.

Lemma 1. *The demand shifters and cost shifters satisfy the following fixed-point equations:*

$$D(a) = (1-\beta)\,\mu^{-\sigma}\int\left(\frac{c(a')}{w}\right)^{\frac{\beta}{1-\beta}(\sigma-1)} D(a')m(a',a)da' + \underbrace{PY}_{GDP} \tag{8}$$

$$c(a) = w^{\beta}\left(\int\left(\mu c(a')\right)^{1-\sigma} m(a',a)da'\right)^{\frac{1-\beta}{1-\sigma}} \tag{9}$$

where $P = \mu\left(\int c(a)^{1-\sigma}dF(a)\right)^{\frac{1}{1-\sigma}}$.

The demand shifter represents the relative size of demand faced by firms at age a relative to the size of the economy, PY, as measured by nominal GDP. The size of demand for intermediate goods is the sum of the portion of aggregate demand $D(a')$ coming from different ages toward a for all a'. The presence of the markup in equation (8) implies that the demand for intermediate goods shrinks as double marginalization is repeated for each intermediate goods transaction. The cost shifters (9) are standard, but the point is that the matching function is a function of a, so the cost function can also be summarized by a.

Using Lemma 1, it is easy to recover the product level equilibrium objects as a function of a. The profit generated by a product with age a can be characterized by

$$\pi(a) = \left(1-\frac{1}{\mu}\right)\left(\frac{\mu c(a)}{P}\right)^{1-\sigma} D(a)$$

The fact that profit is a function of age suggests that the firm's optimization problem can also be summarized in terms of age. A firm with n product lines and age a maximizes the value $V^F(n,a)$

$$r(t)V^F(n,a) = \underbrace{n\pi(a)}_{\text{profits}} - \underbrace{\delta_F n\left\{V^F(n,a) - V^F(n-1,a)\right\}}_{\text{product exit}} + \underbrace{V_a^F(n,a)}_{\text{age effect}}$$
$$+\max_{\lambda\geq 0}\left[\underbrace{n\lambda\left\{V^F(n+1,a) - V^F(n,a)\right\}}_{\text{expansion of variety}} - \underbrace{nw_H\phi(\lambda)}_{\text{R\&D costs}}\right]. \tag{10}$$

In words, the first term on the right-hand side is the total static profit. The second term is the change in firm value due to the exogenous withdrawal of one of its product lines. The third term is the change in firm value due to aging. The fourth term is the change in firm value if a product is added when a new product line arrives with a Poisson arrival rate $n\lambda$. The last term is the R&D cost.

We can show that the value of each firm can be expressed as the sum of the value of the product lines, defined as the net present discounted value of profits from a product line. To show this, guess

$$V^F(n,a) = nV(a)$$

where $V(a)$ is the value of product lines owned by an age a firm, and obtain the following HJB equation for $V(a)$:

$$\left(\rho+\delta_F\right)V(a) = \pi(a) + V_a(a) + \max_{\lambda\geq 0}\left[\lambda V(a) - w_H\phi(\lambda)\right] \tag{11}$$

Finally, the first order condition yields an optimal innovation rate:

$$\lambda(a) = \left\{\frac{\phi}{\gamma w_H}V(a)\right\}^{\frac{1}{\gamma-1}}, \tag{12}$$

which confirms λ is also a function of age.

Kolmogorov Forward Equations

Standard arguments establish that the differential equation governing the evolution of the product density, $f(a)$ at steady state takes the following form:

$$0 = -\frac{\partial f(a)}{\partial a} + (\lambda(a) - \delta_F)f(a) + \lambda_E\boldsymbol{\delta}(a) \tag{13}$$

where $\boldsymbol{\delta}(a)$ denotes the Dirac delta function, which is zero everywhere except if $a = 0$, and satisfies $\int \boldsymbol{\delta}(a)da = 1$. The time evolution of the firm distribution $\frac{\partial f(a)}{\partial t} = 0$ is consistent with the sum of the four terms in the right-hand side: The first term captures the increase in firm age, the second term captures the increase in the number of product lines due to the successful R&D by age a firm minus the exogenous exit of product lines, and the third term captures the addition of product lines due to the entry of age 0 firm.

Labor Market Clearing Conditions

The labor market clearing condition can be expressed using the demand shifters,

$$w = \beta \mu^{-\sigma} \int \left(\frac{c(a)}{P} \right)^{1-\sigma} D(a) f(a) da \tag{14}$$

The high skilled labor market clearing condition is

$$L_H = \int \phi \left(\lambda(a) \right) f(a) da \tag{15}$$

4 R&D Allocation

Having established the decentralized equilibrium in Section 3, we now analyze welfare-maximizing allocations under alternative institutional arrangements. This section presents two benchmark planning problems: the First-Best Social Planner who internalizes all externalities, and the Second-Best Constrained Planner who addresses only innovation externalities while maintaining the decentralized goods market structure. These theoretical benchmarks provide the foundation for our quantitative welfare analysis in Section 5.

4.1 Decentralized Equilibrium R&D Allocation

As a reference point for welfare comparisons, we first characterize the R&D allocation in the decentralized equilibrium established in Section 3.

Proposition 2 (Decentralized Equilibrium R&D Allocation)**.** *The allocation of R&D workers in the decentralized equilibrium satisfies:*

$$l_H(a) = \frac{V(a)^{\frac{\gamma}{\gamma-1}}}{\int V(a)^{\frac{\gamma}{\gamma-1}} f(a) da} \propto V(a)^{\frac{\gamma}{\gamma-1}} \tag{16}$$

where the product value function $V(a)$ solves:

$$\rho V(a) = \underbrace{\pi(a)}_{\text{profit}} \underbrace{-\delta_F V(a)}_{\text{product death}} + \underbrace{V_a(a)}_{\text{age effect}} + \underbrace{\left[\lambda(a)V(a) - w_H \phi(\lambda(a))\right]}_{\text{net value of innovation}} \tag{17}$$

where revenue is $r(a) = \left(\frac{\mu c(a)}{P}\right)^{1-\sigma} D(a)$ *and profits are* $\pi(a) = \left(1 - \frac{1}{\mu}\right) r(a)$.

This allocation is distorted due to both markup pricing in the goods market and the failure to internalize innovation externalities through the production network.

4.2 First-Best Social Planner

The First-Best Social Planner maximizes welfare by internalizing all production and innovation externalities in the economy. Unlike the decentralized equilibrium, the Social Planner directly controls both goods production and R&D allocation to achieve the first-best outcome.

The Social Planner maximizes the discounted utility flow

$$U_0 = \int_0^\infty \exp\left(-\rho t\right) \log Y(t)\, dt \tag{18}$$

where $Y = \left(\int y(a)^{\frac{\sigma-1}{\sigma}} dF(a)\right)^{\frac{\sigma}{\sigma-1}}$ subject to the following constraints: (1), (3), (6), (13), (14), and (15). We use the fact that the allocation of production workers is static to solve a two-stage maximization problem. First, we characterize the allocation of general workers with $f(a)$ and $m(a', a)$ as given. Next, this allocation is solved as a dynamic optimization problem using a maximum value function with respect to $f(a)$ and $m(a', a)$.

First, the social planner's static allocation $\left(y(a), x(a), l(a), x(a', a)\right)$ problem satisfies the following problems:

$$\max \left(\int y(a)^{\frac{\sigma-1}{\sigma}} dF(a)\right)^{\frac{\sigma}{\sigma-1}}$$

subject to

$$x(a)=\frac{1}{\beta^{\beta}\left(1-\beta\right)^{1-\beta}}l\left(a\right)^{\beta}\left(\int x(a',a)^{\frac{\sigma-1}{\sigma}}m(a',a)da'\right)^{\frac{\sigma}{\sigma-1}(1-\beta)}, \tag{19}$$

$$x(a)=\int x(a',a)m(a',a)da'+y(a), \tag{20}$$

$$\int l(a)dF(a)=1 \tag{21}$$

We make the following transformation to compare this problem with the characterization in the cost function in the decentralized equilibrium. By relabeling the Lagrange multipliers of the above problem appropriately, we obtain the following Lemma.

Lemma 2. *The static allocation can be characterized by the following P^{SP}, $D(a)^{SP}$ and $c(a)^{SP}$. They solve the following equations.*

$$D^{SP}(a)=\left(1-\beta\right)\int\left[c(a')^{SP}\right]^{\frac{\beta}{1-\beta}(\sigma-1)}D^{SP}(a')m(a',a)da'+1, \tag{22}$$

$$c^{SP}\left(a\right)=\left(\int\left(c^{SP}(a')\right)^{1-\sigma}m(a',a)da'\right)^{\frac{1-\beta}{1-\sigma}}, \tag{23}$$

$$\left(P^{SP}\right)^{1-\sigma}=\beta\int c^{SP}(a)^{1-\sigma}D^{SP}(a)f(a)da, \tag{24}$$

$$P^{SP}=\left(\int c^{SP}(a)^{1-\sigma}f(a)da\right)^{\frac{1}{1-\sigma}} \tag{25}$$

Given the solutions, we could recover the solution of the original problem ($x^{SP}(a)$, $y^{SP}(a)$, and $l^{SP}(a)$) as $x^{SP}(a)=\frac{D^{SP}(a)}{P^{1-\sigma}c^{SP}(a)^{\sigma}}$, $y^{SP}(a)=c^{SP}(a)^{-\sigma}$, and $l^{SP}(a)=\beta c^{SP}(a)x^{SP}(a)$.

This Lemma 2 is useful for characterizing static allocations because it gives the social planner analogs of the demand (36) and cost shifters (9) and the price index in the decentralized equilibrium. Also, by comparing with Lemma 1, we can see that the two allocations coincide only when $\mu=1$, i.e., the goods market equilibrium is inefficient.

We then redefine the social planner's problem using Lemma 2: using $y^{SP}(a)=c^{SP}(a)^{-\sigma}$, (24), and (25), we transform the social planner's objective function as follows

$$U_0=\int_0^{\infty}\exp\left(-\rho t\right)\log\left(\int c(a)^{1-\sigma}f(a)da\right)dt$$

$$= \int_0^\infty \exp\left(-\rho t\right) \frac{\sigma-1}{\sigma} \log\left(\beta \int c^{SP}(a)^{1-\sigma} D^{SP}(a) f(a) da\right) dt$$

The social planner maximizes the above objective function subject to (6), (13), (22), (23), (24), and (25). The following Proposition characterizes the optimal allocation of the social planner's R&D workers.

Proposition 3 (First-Best Social Planner R&D Allocation)**.** *The allocation of R&D workers in the First-Best Social Planner satisfies:*

$$l_H^{SP}(a) \propto \left(V^{SP}(a)\right)^{\frac{\gamma}{\gamma-1}} \tag{26}$$

where $V^{SP}(a)$ is the social value function that solves:

$$\begin{aligned} \rho V^{SP}(a) = {} & r^{SP}(a) - \delta_F V^{SP}(a) + V_a^{SP}(a) \\ & + \left[\lambda^{SP}(a) V^{SP}(a) - w_H^{SP} \phi(\lambda^{SP}(a))\right] \\ & + \int V^M(a', a) \left\{ \frac{\zeta}{N_f} f(a') - \delta_M m(a', a) \right\} da' \end{aligned} \tag{27}$$

and where $V^M(a', a)$ is the social value of a match between a buyer of age a and a supplier of age a'. The term w_H^{SP} is the shadow wage on R&D labor in the planner's problem.

By comparing the value functions of Propositions 2 and 3, we can identify two channels of R&D misallocation.

Two Channels of R&D Misallocation

Distortions from Market Power: The first channel arises from markup pricing in the goods market. While the markup μ is uniform across firms, the network structure creates *heterogeneous degrees of double marginalization*. Older, more central firms are embedded in longer supply chains, causing their revenues, $r(a)$, to be disproportionately suppressed relative to the social planner's equivalent, $r^{SP}(a)$. This distorts the private value of innovation, $V(a)$, and consequently the allocation of R&D resources. This force, related to the static misallocation literature, systematically disincentivizes R&D in older firms.

Network-Formation Externalities: The second channel is a network-formation externality. The fundamental source of this externality is that an individual firm a's R&D investment, $\lambda(a)$, alters an aggregate state variable of the economy—the product density $f(a)$—an effect the firm does not internalize. A change in $f(a)$ affects the matching process for all other firms, altering the composition of trading opportunities. Specifically, when a relationship is formed or dissolved, a social surplus, measured by the value $V^M(a', a)$, is created or destroyed. However, firm a only considers its own private portion of this value, ignoring the benefit accruing to or the loss incurred by its supplier, a'. This uninternalized value is quantified as the "network-formation externality wedge":

$$\text{wedge}(a) = \int V^M(a', a) \left\{ \frac{\zeta}{N_f} f(a') - \delta_M m(a', a) \right\} da' \tag{28}$$

This wedge evaluates the marginal contribution of a firm of type a's existence to the net flow of relationships in the market, valued at the full social value V^M. The magnitude of this externality varies with firm age a, leading to a misallocation of R&D resources. A quantitative assessment of the resulting misallocation is conducted in Section 5.

4.3 Second-Best Constrained Planner

To isolate the pure effect of innovation externalities and understand their contribution to total welfare losses, we introduce the Second-Best Constrained Planner. This planner represents an intermediate case between the decentralized equilibrium and the First-Best Social Planner, allowing us to decompose welfare distortions into two distinct channels: markup distortions in goods markets and innovation externalities in R&D allocation. By maintaining the decentralized goods market structure while optimally allocating R&D workers, the Constrained Planner addresses only innovation externalities, enabling us to quantify the isolated welfare impact of network formation externalities.

The Constrained Planner maximizes the same objective function (18) as the First-Best Social Planner:

$$U_0^{CP} = \int_0^\infty \exp(-\rho t) \log Y(t)\, dt \tag{29}$$

subject to maintaining the decentralized goods market structure from Section 3: (6), (8), (9), (13), (14), and (15). The Constrained Planner optimally allocates only R&D workers $l_H(a)$ to address innovation externalities while preserving the static goods market distortions characterized by the decentralized markup structure.

Proposition 4 (Constrained Planner's R&D Allocation)**.** *The Constrained Planner's allocation of R&D workers satisfies:*

$$l_H^{CP}(a) \propto [V^{CP}(a)]^{\frac{\gamma}{\gamma-1}} \tag{30}$$

where the constrained social value function $V^{CP}(a)$ solves:

$$\begin{aligned}\rho V^{CP}(a) = {} & r(a) - \delta_F V^{CP}(a) + V_a^{CP}(a) \\ & + \left[\lambda^{CP}(a) V^{CP}(a) - w_H^{CP} \phi(\lambda^{CP}(a))\right] \\ & + \int V^{M,CP}(a', a) \left\{ \frac{\zeta}{N_f} f(a') - \delta_M m(a', a) \right\} da' \end{aligned} \tag{31}$$

where $r(a) = \left(\frac{\mu c(a)}{P}\right)^{1-\sigma} D(a)$ uses the decentralized demand and cost shifters $D(a)$ and $c(a)$ from equations (8) and (9), respectively, and w_H^{CP} denotes the corresponding shadow wage on R&D labor.

The Constrained Planner represents a realistic policy scenario where governments can influence R&D allocation through subsidies or taxes, but cannot directly intervene in goods market pricing decisions due to informational or institutional constraints.

Elimination of Intensive Margin Distortions

By maintaining the decentralized goods market structure, the Constrained Planner eliminates the intensive margin distortions present in the First-Best Social Planner. The markup distortions from equations (8) and (9) are preserved, meaning that the heterogeneous double marginalization effects remain unchanged from the decentralized equilibrium.

Consequently, the Constrained Planner addresses only the extensive margin externalities—the network formation externalities captured by the $\int V^{M,CP}(a', a) \left\{ \frac{\zeta}{N_f} f(a') - \delta_M m(a', a) \right\} da'$ term in equation (31). This isolates the pure effect of internalizing network formation externalities while maintaining all markup-related distortions, making it particularly relevant for understanding the welfare gains from targeted R&D policies.

5 Quantitative Analysis

This section presents the estimation of our supply chain innovation model with network-formation externalities using Japanese firm-level data, providing empirical validation of the theoretical framework developed in Section 4. We employ a two-stage approach: first, we estimate six structural parameters using Simulated Method of Moments (SMM) under the decentralized equilibrium characterized by Proposition 2, targeting moments from the observed age-based matching matrix. Second, we use these estimated parameters as inputs to solve for counterfactual allocations under the alternative institutional arrangements analyzed in Section 4—specifically the First-Best Social Planner and Second-Best Constrained Planner solutions from Propositions 3 and 4.

This methodology allows us to quantify the welfare implications of the two channels of R&D misallocation identified in Section 4: distortions from market power and network formation externalities. By comparing the estimated decentralized equilibrium with the theoretical benchmarks, we can isolate the welfare effects from addressing different types of market failures while maintaining structural consistency across all policy environments.

5.1 Parameter Estimation

5.1.1 Estimation Strategy

Our estimation strategy follows a two-stage approach designed to isolate the welfare effects of policy interventions. In the first stage, we estimate structural parameters under the decentralized equilibrium using Simulated Method of Moments (SMM), targeting moments from the age-based matching matrix that characterizes network formation patterns in Japanese supply chains. The model is estimated using 100 Halton draws to reduce simulation variance, with the objective function minimizing the weighted distance between empirical and model-implied moments.

The parameter vector to be estimated is $\boldsymbol{\theta} = (\zeta, \zeta_0, \delta_M, \gamma, \phi, \delta_F)$, where ζ represents the network formation efficiency parameter, ζ_0 captures the initial network density at firm entry, δ_M is the network link destruction rate, γ determines the R&D cost curvature, ϕ measures R&D productivity, and δ_F represents the firm exit rate.

In the second stage, we take the estimated parameter vector $\hat{\boldsymbol{\theta}}$ as given and solve for equilibrium allocations under the alternative institutional arrangements analyzed in Section 4. This approach ensures that all welfare comparisons are conducted under identical structural parameters, isolating the pure effect of policy interventions. The First-Best Social Planner solution from equation (26) represents the first-best allocation that internalizes all externalities, while the Second-Best Constrained Planner from equation (30) addresses only the network formation externalities while preserving the distortions from market power present in the decentralized market structure.

5.1.2 Estimated Parameters

The following parameters were estimated using Simulated Method of Moments targeting moments from the age-based matching matrix that characterizes network formation patterns in Japanese supply chains.

The network formation rate $\zeta = 0.3165$ indicates moderate efficiency in supply chain matching, while the high initial connectivity $\zeta_0 = 0.7371$ suggests new firms enter with

Table 3: Model Parameters

Parameter	Symbol	Value
Internally Estimated Parameters		
Network formation rate	ζ	0.3165
Initial connectivity	ζ_0	0.7371
Link destruction rate	δ_M	0.0100
R&D curvature	γ	2.1316
R&D efficiency	ϕ	0.0003842
Firm exit rate	δ_F	0.0400
Externally Calibrated Parameters		
Elasticity of substitution	σ	3
Labor share	β	0.33
Discount rate	ρ	0.05
Entry mass	λ_E	1

Notes: Internal parameters estimated using Simulated Method of Moments targeting 16 moments from the age-based matching matrix M_{aa}. SMM estimation uses 100 Halton draws to reduce simulation variance.

substantial existing networks.

The innovation parameters reveal substantial convexity in R&D costs ($\gamma = 2.13$) and low base productivity ($\phi = 0.0004$), consistent with high-risk, high-return industrial innovation. The labor share parameter $\beta = 0.33$ captures the double marginalization structure in which upstream suppliers receive one-third of value added while downstream firms retain two-thirds.

The externally calibrated parameters are set based on previous literature and institutional knowledge. The elasticity of substitution $\sigma = 3$ follows Miyauchi (2024), representing moderate demand substitutability between intermediate goods. The discount rate $\rho = 0.05$ reflects a standard value commonly used in the literature for firm-level time preferences. The entry mass $\lambda_E = 1$ serves as a normalization of the firm entry flow. Under monopolistic competition with constant elasticity of substitution, the implied markup is $\mu = \sigma/(\sigma - 1) = 1.5$, capturing the price–cost margin that emerges from optimal pricing behavior.

5.1.3 Data Construction and Age Ranking

Before presenting model fit results, we describe the construction of the empirical matching matrix used in estimation. Firms are classified into age quantiles rather than continuous age values to ensure sufficient observations in each cell and to facilitate comparison with model predictions.

The empirical matching matrix is constructed as follows: First, firms are ranked into

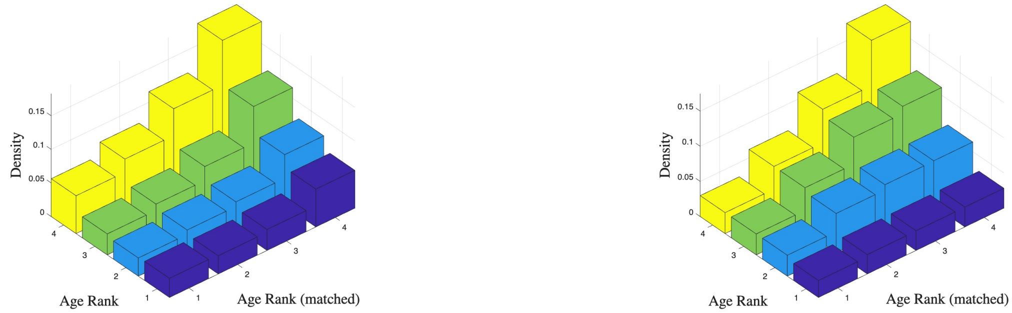


Figure 5: Model Fit: Empirical (left) vs Simulated (right) Matching Matrices

Notes: The 3D surfaces show the density of network connections between firms of different age categories.

age quantiles based on their founding year, creating discrete age categories that ensure balanced sample sizes across groups. Second, for each firm pair connected in the supply chain network, we record both buyer and seller connections and compute the average connection strength between age groups. Third, the resulting density matrix is normalized so that all entries sum to one, representing the probability distribution of connections across age pairs.

5.1.4 Model Fit

Figure 5 presents the model's fit to the targeted moments through side-by-side 3D visualizations of the empirical and simulated matching matrices. The model successfully captures the age-based network formation patterns, with some systematic deviations for very young and very old firms.

The model achieves a reasonable fit for most moments, capturing the essential features documented in Section 2 while exhibiting some systematic deviations. The empirical analysis reveals several patterns that the model successfully replicates: (1) the number of suppliers and buyers increases monotonically with firm age, but with decreasing rates of growth as firms mature; (2) firms exhibit positive age-assortative matching, where older firms preferentially connect with other older firms; and (3) business relationships display substantial stickiness, with more than 10 years required for 90% of relationships to dissolve.

The model successfully captures these age-dependent network formation patterns. The positive age-assortative matching observed in the data emerges naturally from the combination of random matching and relationship stickiness. As documented in Section 2, while new relationship formation shows little age dependence, the persistence of existing relationships causes firms and their partners to "age together," creating the observed assortativity.

The lifecycle patterns of innovation activity are also well-replicated. Section 2 documents that both patent counts and R&D/sales ratios are monotonically increasing functions of firm age, consistent with the model's prediction that innovation incentives grow with network connectivity. The empirical finding that connected firms' R&D activities positively affect their partners' innovation provides validation for the network-formation externality mechanisms central to the model.

However, some model limitations emerge in the precise quantitative fit. The random matching assumption may not fully capture the complex partner selection processes observed in reality, particularly for very young firms who may have access to specialized network formation channels. Additionally, the model's assumption of symmetric relationship formation may oversimplify the heterogeneous search and matching costs that vary across firm ages and industries.

To further verify that our model captures the empirical patterns documented in Section 2, we examine the targeted moments from Figure 5 from different perspectives. Figure 6 presents an alternative visualization of the age-based matching patterns, confirming that the model successfully replicates the positive age assortativity observed in Section 2 data, where firms tend to form partnerships with other firms of similar ages. While the model exhibits slightly younger firm ages compared to Section 2 data, it successfully captures the main empirical patterns, including the age assortativity structure and matching density distribution observed in Japanese supply chain networks.

Figure 7 provides another perspective on the matching patterns, examining how network connectivity and production efficiency evolve jointly over the firm lifecycle. The left panel shows the mass of connections and the right panel shows the growth rate of connections. This pattern confirms the Section 2 finding that mature firms achieve higher productivity through extensive supplier relationships, with both the mass and growth rate of connections exhibiting the tendencies documented in the empirical analysis.

5.2 Lifecycle Dynamics and R&D Misallocation

Our quantitative model, consistent with the empirical evidence in Section 2, generates systematic patterns in how network positions and innovation incentives co-evolve over the firm lifecycle. These dynamics are the key to understanding the origins and nature of R&D misallocation.

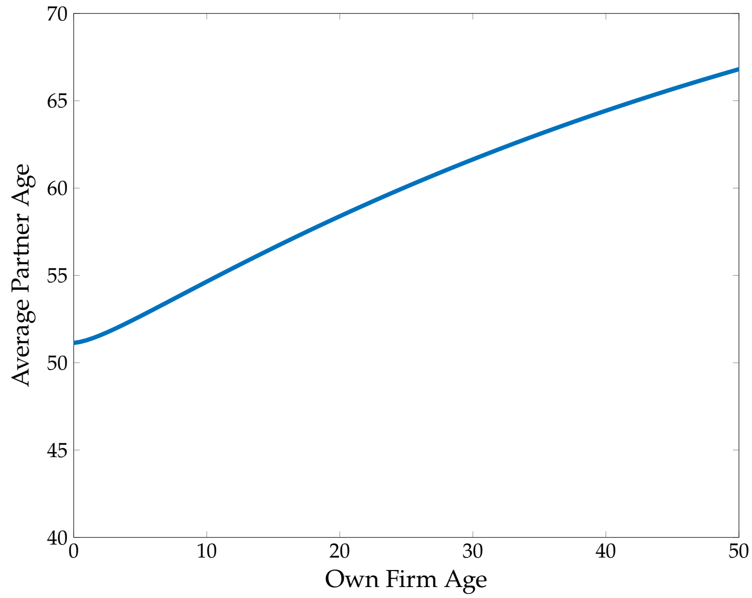


Figure 6: Age Assortativity in Business Networks

Notes: Average partner age plotted against a firm's own age. The 45-degree line represents perfect age assortativity.

5.2.1 Network Formation and Innovation Over the Lifecycle

Beyond the matching moments targeted in estimation, the model generates lifecycle patterns for innovation and productivity that align with the data. Figure 8 illustrates that innovation rates rise with age as firms build larger and more valuable supplier networks. This creates a powerful complementarity between a firm's network position and its incentive to innovate. As firms age, the model's dynamic process of random matching and link stickiness leads them to accumulate a larger mass of connections, as documented in Figure 1a. The emergent network structure produces the positive age-assortative matching shown in Figure 6, where older firms disproportionately connect with other mature firms. This pattern drives two key mechanisms underlying higher productivity among older firms: first, the *love of variety effect* from accessing a broader range of intermediate inputs, and second, the *quality composition effect* from preferentially matching with more productive mature suppliers. These complementarities between network position and innovation create the crucial backdrop against which R&D decisions and their allocative consequences unfold.

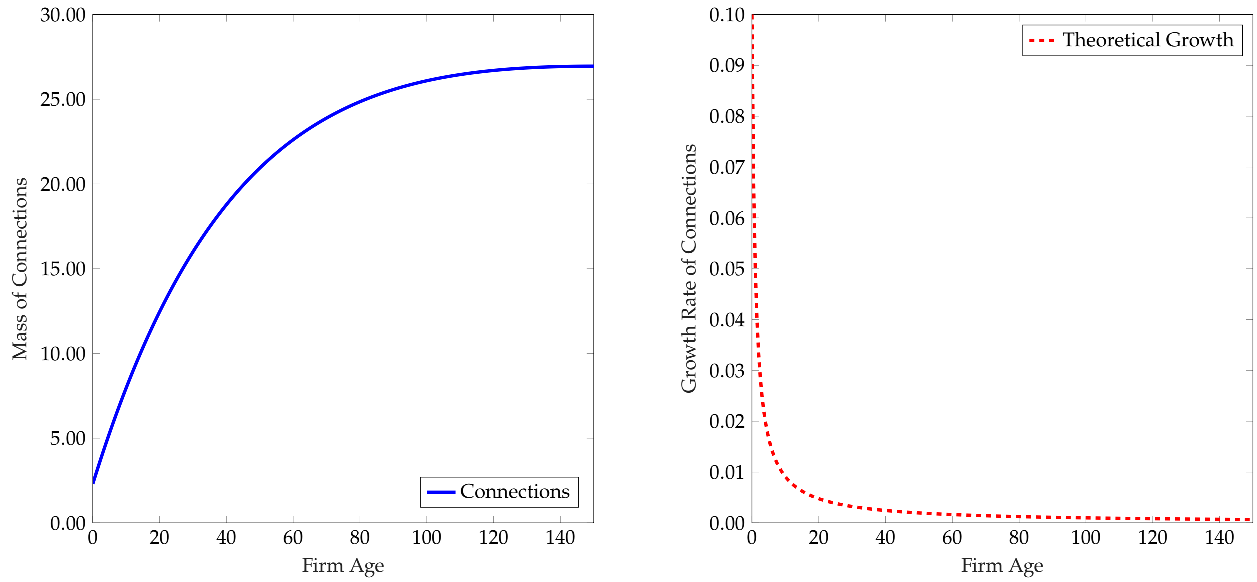


Figure 7: Lifecycle Production Network Analysis

Notes: Left panel shows mass of connections by firm age. Right panel shows growth rate of connections by firm age.

Table 4: Welfare Outcomes

Environment	Equilibrium	Social Planner	Constrained Planner
Welfare gain	–	35.84%	0.09%

5.2.2 Welfare Analysis and Sources of R&D Misallocation

Figure 9 reveals a striking pattern: older firms occupy central positions in production networks with more buyers and suppliers, generating higher sales levels, yet their revenues remain disproportionately suppressed relative to the socially optimal level due to markup distortions that compound through supply chains. This suppression becomes more pronounced with age, reflecting how the cumulative effect of double marginalization particularly penalizes firms embedded in longer supply chains.

To analyze the welfare implications of this structure, we compare three policy environments as defined in Section 4: the *decentralized equilibrium* (DE), the *social planner* (SP), and the *constrained planner* (CP).

Table 4 presents the welfare outcomes, while Figures 10a and 10b show the corresponding R&D allocations. A key observation from Figure 10b is that the SP's optimal R&D allocation is remarkably similar to that of the DE. This finding allows us to infer the primary source of the large welfare gain reported in Table 4. Since the R&D allocation barely changes, the substantial 35.84% gain under the SP must predominantly stem from

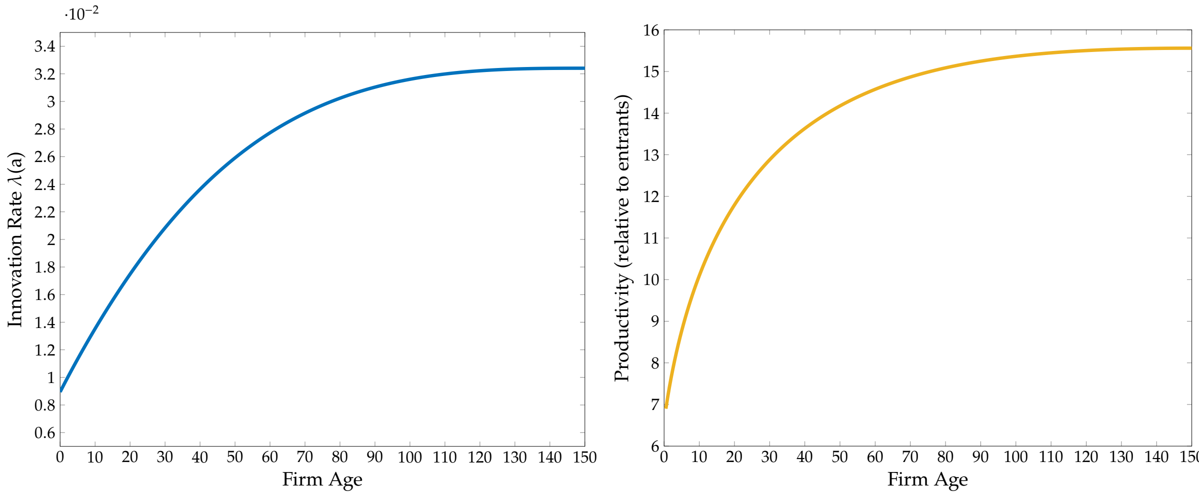


Figure 8: Firm Lifecycle Dynamics

Notes: Innovation rates increase with age (left panel), leading to productivity accumulation over the firm lifecycle (right panel). Productivity is defined as the inverse of marginal cost, $q(a) = 1/c(a)$, normalized relative to entrants. The patterns reflect the complementarity between network position and innovation incentives.

correcting the *misallocation of production inputs* (general labor) that arises from eliminating markup distortions. The minimal welfare gain of 0.09% under the CP, where only R&D resources are reallocated, corroborates this conclusion.

This raises the central question for our analysis: if significant network-formation externalities are present, why does the SP's optimal R&D allocation so closely track the distorted decentralized outcome, especially when the CP allocation deviates significantly? The answer lies in the interaction of two powerful, counteracting forces that shape the SP's allocation.

The first force arises from the *correction of market power distortions*. In the DE, double marginalization distorts the allocation of both production labor and R&D labor by disproportionately suppressing the revenue and private value of innovation, $V(a)$, for older firms embedded in long supply chains. The SP, by eliminating markups as part of the transition from Proposition 2 to 3, boosts the relative value of these older firms. This effect, in isolation, creates a strong incentive to reallocate R&D resources *towards* older firms.

The second force is the *internalization of network-formation externalities*. This effect is isolated by the CP, whose optimal allocation is governed by Proposition 4. This allocation differs from the DE only by the inclusion of the network-formation externality wedge from

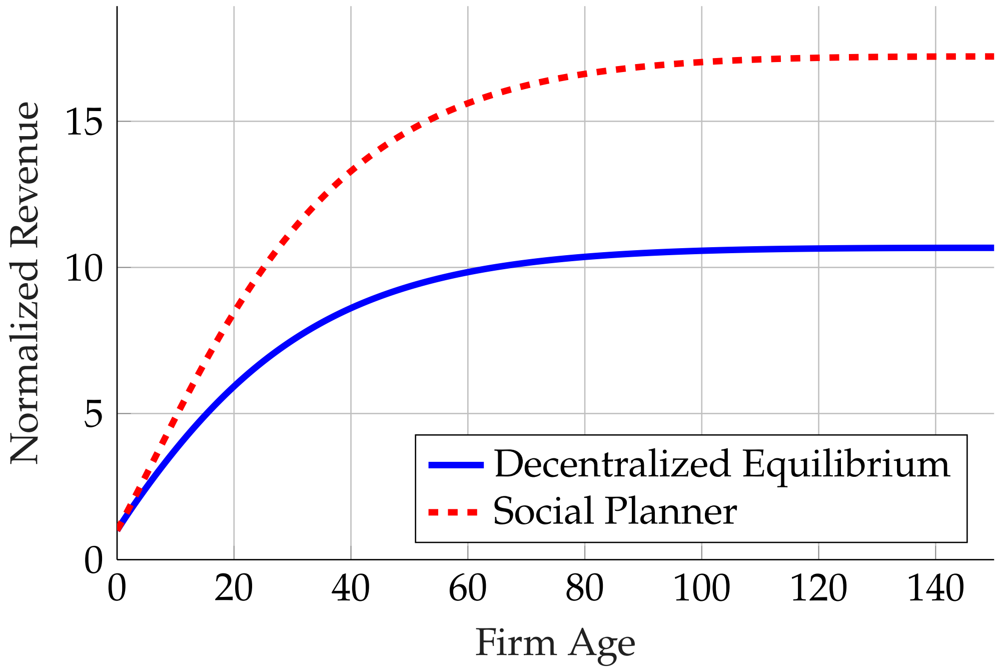


Figure 9: Revenue Comparison by Firm Age

Notes: Revenue patterns across firm age under Decentralized Equilibrium (DE) and Social Planner (SP) allocations. Both series are normalized to 1 at age 0.

equation (28):

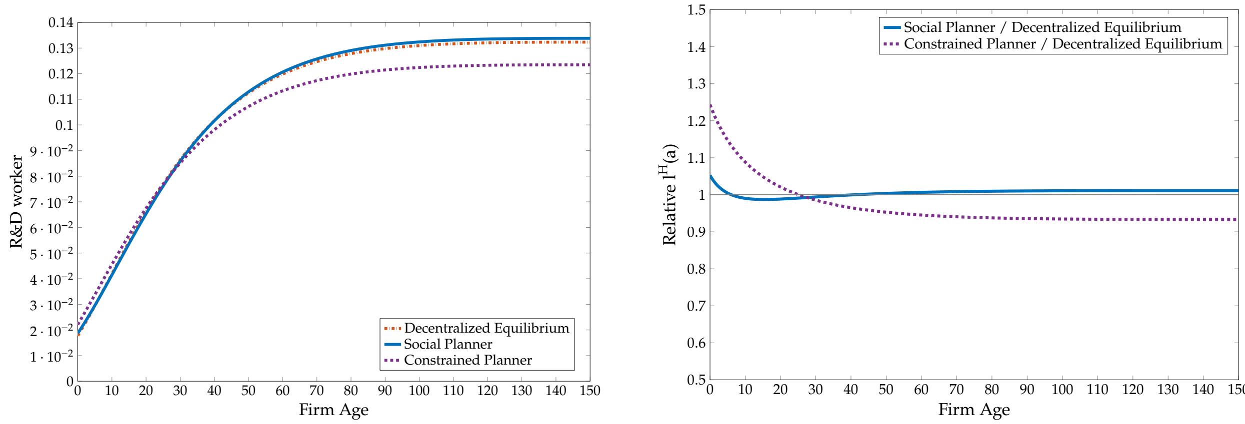
$$\text{wedge}(a) = \int_0^\infty V^M(a', a) \left[\underbrace{\frac{\zeta f(a')}{N_f}}_{\text{Formation}} - \underbrace{\delta_M m(a', a)}_{\text{Destruction}} \right] da'$$

Here, $V^M(a', a)$ is the total social value generated by the trading relationship between a buyer of age a and a supplier of age a'. This value represents the discounted present value of all future surpluses created by the link, and is determined by its own Hamilton-Jacobi-Bellman equation from Proposition 3:

$$\left(\rho + \delta_F + \delta_M - \lambda(a)\right) V^M(a', a) = V^M_a(a', a) + V^M_{a'}(a', a) + \Omega(a', a) r^{SP}(a)$$

where $\Omega(a', a) = \frac{c(a')^{1-\sigma}}{\int c(a'')^{1-\sigma} m(a'', a') da''}$ is the expenditure share of supplier a' in buyer a's total intermediate consumption. The value of a match is thus driven by the flow of revenue, $r^{SP}(a)$, generated through the link.

The wedge thus quantifies the value a firm of age a fails to internalize at the two margins



(a) R&D workers' allocation

(b) R&D allocation distortions

Figure 10: R&D allocation outcomes across policy environments

Notes: The left panel shows normalized R&D labor demand across the three policy environments. The right panel shows the relative allocation ratios to the decentralized equilibrium.

of relationship dynamics. The first term inside the integral is the formation component, representing the gross inflow of new relationships. When a match occurs, a social surplus $V^M(a', a)$ is created, but the innovating firm a neglects the portion accruing to its new supplier a'. The second term is the destruction component, representing the gross outflow of dissolving relationships. When a link is lost, the social surplus $V^M(a', a)$ is destroyed, but firm a externalizes the loss incurred by its supplier.

As shown by the dotted line in Figure 10b, internalizing this wedge alone, as the CP does, reallocates R&D resources *away* from the oldest firms relative to the DE. This direction results from the interplay of two competing effects. The first is the *match value effect*: since older buyer firms are more productive, the social value of a single match with them, $V^M(a', a)$, is higher. This effect, in isolation, would make the externality larger for older firms. The second is the *net flow effect*: young firms, with few existing relationships to lose, have a much higher net rate of relationship formation. Our quantitative results show that this second effect dominates. The value per match, V^M, does not increase steeply enough with age to offset the rapid decline in the net formation rate. Consequently, the uninternalized value is largest for young firms, leading the CP to reallocate R&D resources toward them.

In the Social Planner's solution, these two forces act simultaneously and in opposite directions on older firms. The push to allocate *more* resources to them to correct for market power distortions is largely offset by the push to allocate *less* to them due to the specifics of the network-formation externality. This cancellation explains why the SP's final R&D allocation appears close to the original decentralized one. The CP's environment, lacking

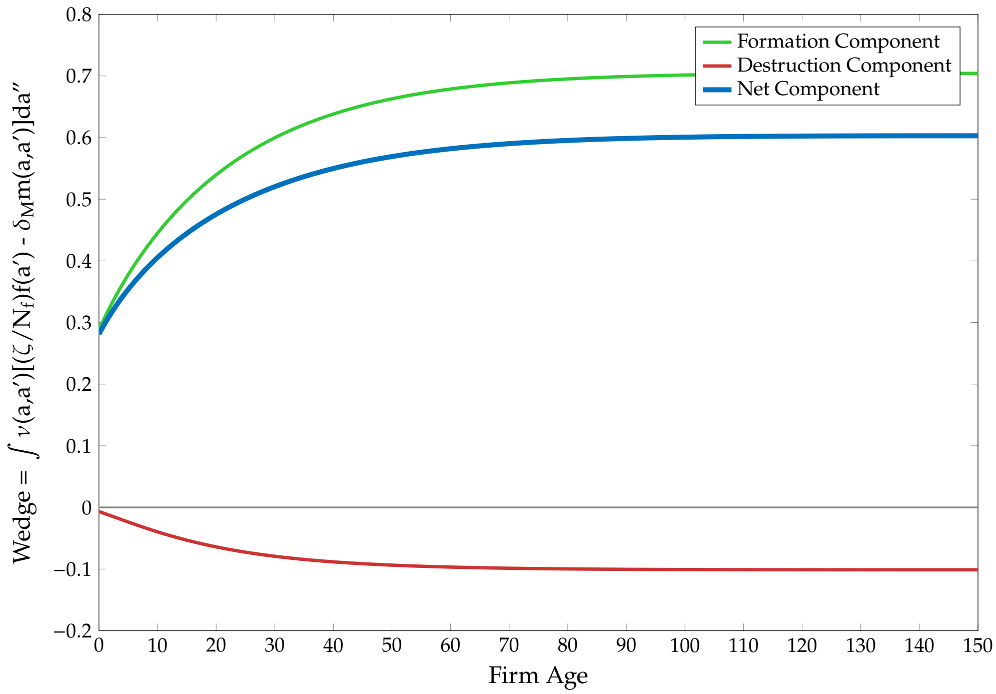


Figure 11: Network-Formation Externality Wedge Decomposition

Notes: Decomposition of the network-formation externality wedge from equation (28) computed under the Constrained Planner equilibrium. The formation component represents $\int V^M(a', a)\frac{\zeta f(a')}{N_f}da'$ and the destruction component represents $\int V^M(a', a)\delta_M m(a', a)da'$. The net component is their difference.

the first force, reveals the unmitigated impact of the second, explaining its significant deviation from the DE.

5.2.3 Policy Implications

Our quantitative results offer a clear policy implication, though one with important trade-offs. The minimal welfare gain under the Constrained Planner (0.09%) demonstrates that correcting R&D misallocation in isolation yields negligible benefits. This suggests that the optimal policy focus should be on addressing the large welfare losses stemming from the misallocation of production inputs due to market power.

However, a policy focused solely on eliminating market power could have a counterintuitive side effect on the allocation of R&D. As the Constrained Planner's solution reveals, correcting only for the network-formation externality pushes the R&D allocation for older firms even further from the decentralized equilibrium than the first-best allocation does (Figure 10b). This implies that the optimal policy of removing markups might be accompanied by an amplification of the R&D misallocation. Nevertheless, our findings suggest this is a worthwhile trade-off. The quantitative impact of the R&D resource misallocation on aggregate welfare is vanishingly small compared to the large, first-order gains from correcting the misallocation of production inputs.

6 Conclusion

This paper investigates how production networks shape firms' R&D decisions, identifying and quantifying the aggregate inefficiencies that arise from this relationship. Guided by empirical patterns from Japanese firm-level data, we build a dynamic model where firms establish supplier networks through an exogenous matching process and leverage these connections to introduce new products.

Our framework uncovers a novel *network-formation externality*: firms' private R&D decisions improve the market-wide pool of trading partners, a positive spillover for which they are not compensated. When we take this model to the data, our central finding is that this externality interacts with market power distortions in a crucial way. The optimal R&D allocation in a first-best world is nearly identical to the decentralized allocation. This is because two counteracting forces on older firms almost perfectly offset each other: correcting for markups pushes R&D resources towards them, while internalizing the network-formation externality pushes resources away.

This result leads to a clear policy conclusion. Because the forces governing R&D allocation cancel out, the optimal policy derives its substantial welfare gains not from reallocating R&D, but from correcting the misallocation of *production inputs* due to market power. Our findings underscore the importance of understanding the general equilibrium interactions between market failures. Policies that target dynamic inefficiencies like R&D misallocation in isolation, without addressing the large, underlying distortions from market power, may have limited impact on aggregate welfare.

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Appendix for "Production Networks and R&D Allocation"

A Additional Empirical Results

A.1 Cross-sectional correlation between production and citation networks

In this section, we show that the production and citation networks overlap at the firm level. Using the combined TDB and IIP patent databases, we find that citation relationships in Japan are strongly associated with production links. Conditional on an observed transactional relationship, the probability of a citation tie is roughly 30 times larger than under a random benchmark. Moreover, this probability declines monotonically with network distance, underscoring the importance of distinguishing the two networks when working with firm-level data.

The integrated database covers Japanese firms that filed at least one patent application between 1998 and 2019. We construct a patent-citation matrix with citing firms in rows and cited firms in columns, where an entry of one indicates the presence of a citation link. Analogously, we build an input-output matrix that records inter-firm production relationships among patenting firms; the columns index suppliers, the rows index purchasers, and a value of one denotes an active business relationship. These matrices enumerate the potential links between Japanese patenting firms, which we use to study how production and patent networks interact.

Panel (a) summarizes the estimates from a linear probability model relating direct production links to patent citations. Columns (1)–(4) vary the set of control variables. The estimated coefficient implies that, conditional on a direct business relationship, the probability of observing a citation link rises by about 35% relative to the random benchmark.

The right column of panel (a) examines higher-order production-network linkages. The probability of a citation link declines steadily with network distance: second- and third-order links remain relevant, whereas fourth-order and more distant connections are nearly irrelevant. Panel (b) provides a visual summary of this decay with respect to production-network distance.

Table A1: Summary of Production and Patent Network Links

Number of Patent Firms	10,481	
Possible Combinations of Firms	109,851,361	
Active Patent Filing Links	2,548,881	(2.3%)
Active Production Network Links	1,647,770	(1.5%)

Alternative measurement

To address concerns about the parametric assumptions of the linear probability model, we also compute conditional probabilities for different subsamples $g \in G$,

$$P_C \equiv \Pr(\text{patent connection} = 1 \mid \text{network connection} = 1, g).$$

We estimate them empirically as

$$\hat{P}_C = \frac{|\text{patent connection} = 1 \cap \text{network connection} = 1, g|}{|\text{network connection} = 1, g|},$$

and compare them with the corresponding unconditional probabilities,

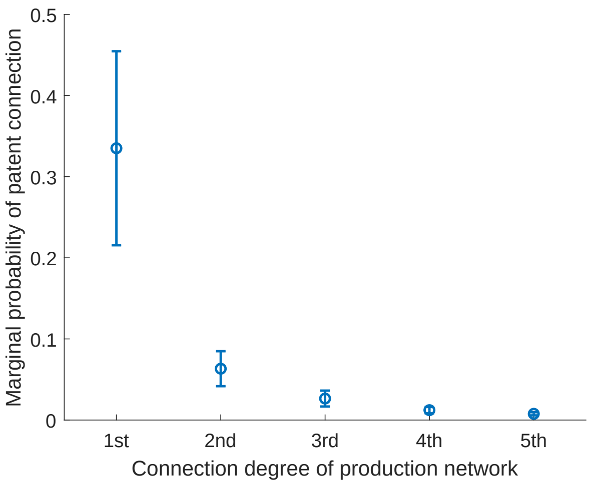
$$\hat{P}_U = \frac{|\text{patent connection} = 1, g|}{|g|}.$$

We form subsamples using every combination of firm-pair characteristics—industry, firm-size decile, and location. Figure A1 plots the conditional probabilities against the corresponding unconditional probabilities for each group.

Across all groupings, the unconditional probability is close to zero because the matrices are very sparse, whereas the average conditional probability remains around 30–40% given a production-network connection.

Table A2: Production Network and Patent Connection

(a) Network degrees and patent connection



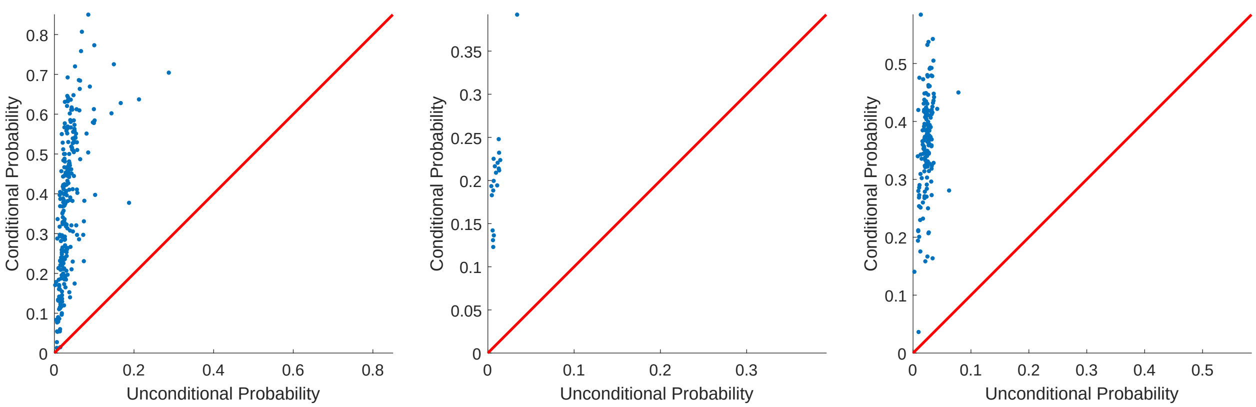
Notes:

	First Degree Connection				Higher Order Degree Connection			
	(1)	(2)	(3)	(4)	Second	Third	Fourth	Fifth
network connection	0.343***	0.339***	0.344***	0.335***	0.063***	0.026***	0.012***	0.007***
	(0.062)	(0.062)	(0.062)	(0.061)	(0.011)	(0.005)	(0.002)	(0.001)
Industry Pair FE	✓			✓	✓	✓	✓	✓
Size Pair FE		✓		✓	✓	✓	✓	✓
Prefecture Pair FE			✓	✓	✓	✓	✓	✓
Observations	93161103	93161104	93161104	93161103	93161103	93161103	93161103	93161103

Standard errors in parentheses

* p<0.10, ** p<0.05, *** p<0.01

Panel (b) reports linear probability estimates that relate the possible combinations of patent firm-to-firm networks in Table A1 to the presence of production-network links, controlling for firm-pair characteristics: (a) two-digit Japanese Standard Industrial Classification, (b) firm-size deciles, and (c) prefectures. Panel (a) plots the corresponding coefficients.



(a) By industry pairs (b) By firm size (deciles) pairs (c) By location pairs

Figure A1: Conditional and unconditional probability of patent connection.

Notes: The x-axis reports the unconditional probability $\frac{|\text{patent connection}=1,g|}{|g|}$, and the y-axis shows the conditional probability $\frac{|\text{patent connection}=1 \cap \text{network connection}=1,g|}{|\text{network connection}=1,g|}$. The groups $g \in G$ include all combinations of firm-pair characteristics: (a) two-digit Japan Standard Industrial Classification, (b) firm-size deciles, and (c) prefectures. Groups with fewer than 50 observations in either patent relationships or production-network connections are excluded.

B Equation list of Equilibrium

The list of equations for solving the steady state equilibrium is as follows: (i) the value of product $V(a)$; (ii) the innovation rates $\lambda(a)$; (iii) the distribution of products $f(a)$; (vi) the distribution of matched buyers and suppliers $m(a', a)$; (v) the cost function $c(a)$; (vi) the demand shifter $D(a)$; (vii) the wage for production worker w; (viii) the wage for R&D worker w_H, and (ix) the price index P

1. The value of products:

$$\left(\rho + \delta_F - \lambda(a)\right) V(a) = \left(1 - \frac{1}{\mu}\right) r(a) + V_a(a) - w_H \phi(\lambda(a))$$

   where revenue, $r(a)$ is

$$r(a) = \left(\frac{\mu c(a)}{P}\right)^{1-\sigma} D(a)$$

   and the price index, P is

$$P = \mu \left(\int c(a)^{1-\sigma} dF(a)\right)^{\frac{1}{1-\sigma}}$$

2. FOC for the innovation rate:

$$\lambda(a) = \left\{\frac{\phi}{\gamma w_H} V(a)\right\}^{\frac{1}{\gamma-1}} \tag{32}$$

3. The distribution of products $f(a)$:

$$0 = -\frac{\partial}{\partial a} f(a) + (\lambda(a) - \delta_F) f(a) + \lambda_E \boldsymbol{\delta}(a) \tag{33}$$

4. The distributions of matched products $m(a', a)$:

$$\frac{\partial}{\partial a} m(a', a) = -\frac{\partial}{\partial a'} m(a', a) - (\delta_M + \delta_F) m(a', a) + \frac{\zeta}{N_f} f(a') + \zeta_0 \frac{\lambda_E}{N_f} \boldsymbol{\delta}(a) + \lambda(a') m(a', a) \tag{34}$$

   subject to the boundary condition $m(a'; 0) = \zeta_0 f(a')$.

5. The cost shifter $c\left(a\right)$ satisfies

$$c\left(a\right)=w^{\beta}\left(\int\left(\mu c(a')\right)^{1-\sigma}m(a',a)da'\right)^{\frac{1-\beta}{1-\sigma}} \tag{35}$$

where $w=\frac{\beta}{\mu-\left(1-\beta\right)}$.

6. The demand shifter for an age a product $D(a)$ satisfies

$$D(a)=\left(1-\beta\right)\mu^{-\sigma}\int\left[\frac{c(a')}{w}\right]^{\frac{\beta}{1-\beta}(\sigma-1)}D(a')m(a',a)da'+1 \tag{36}$$

7. The skilled wage w_H satisfies the skilled labor market clearing condition

$$1=\int\phi\left(\lambda(a)\right)f(a)da \tag{37}$$

Equation list of social planner solution:

1. Social value function

$$\begin{aligned}\left(\rho+\delta_F-\lambda^{SP}(a)\right)V^{SP}(a)=&r^{SP}(a)+\frac{\partial}{\partial a}V^{SP}(a)-w_H^{SP}\phi\left(\lambda^{SP}(a)\right)\\&+\int V^{M}\left(a',a\right)\left\{\frac{\zeta}{N_f}f(a')-\delta_M m(a',a)\right\}da',\end{aligned}$$

where

$$r^{SP}(a)=\left(\frac{c^{SP}(a)}{P^{SP}}\right)^{1-\sigma}D^{SP}(a)$$

$$P^{SP}=\left(\int c^{SP}(a)^{1-\sigma}f(a)da\right)^{\frac{1}{1-\sigma}}$$

$$\begin{aligned}\left(\rho+\delta_F+\delta_M-\lambda^{SP}\left(a\right)\right)V^{M}\left(a',a\right)=&\frac{\partial}{\partial a}V^{M}\left(a',a\right)+\frac{\partial}{\partial a'}V^{M}\left(a',a\right)\\&+r^{SP}(a)\left(1-\beta\right)\Omega(a',a)f(a')\end{aligned}$$

$$\Omega(a',a)=\frac{c^{SP}(a')^{1-\sigma}}{\int c^{SP}(a'')^{1-\sigma}m(a'',a)da''}$$

2. FOC for the innovation rate:

$$\lambda^{SP}(a) = \left\{ \frac{\phi}{\gamma w_H^{SP}} V^{SP}(a) \right\}^{\frac{1}{\gamma-1}} \tag{38}$$

3. The distribution of products $f(a)$:

$$0 = -\frac{\partial}{\partial a} f(a) + \left(\lambda^{SP}(a) - \delta_F\right) f(a) + \lambda_E \boldsymbol{\delta}(a) \tag{39}$$

4. The distribution of matched products $m(a', a)$:

$$\begin{aligned} \frac{\partial}{\partial a} m(a', a) = &- \frac{\partial}{\partial a'} m(a', a) - (\delta_F + \delta_M) m(a', a) + \frac{\zeta}{N_f} f(a') \\ &+ \zeta_0 \frac{\lambda_E}{N_f} \boldsymbol{\delta}(a) + \lambda^{SP}(a') m(a', a) \end{aligned} \tag{40}$$

subject to the boundary condition $m(a'; 0) = \frac{\zeta_0}{N_f} f(a')$.

5. The cost shifter $c(a)$ satisfies

$$c^{SP}(a) = \left(\int \left(c^{SP}(a') \right)^{1-\sigma} m(a', a) da' \right)^{\frac{1-\beta}{1-\sigma}} \tag{41}$$

6. The demand shifter for an age a product $D(a)$ satisfies

$$D^{SP}(a) = (1 - \beta) \int c(a')^{SP \frac{\beta}{1-\beta}(\sigma-1)} D^{SP}(a') m(a', a) da' + 1 \tag{42}$$

7. The skilled wage w_H^{SP} satisfies the skilled labor market clearing condition

$$1 = \int \phi \left(\lambda^{SP}(a) \right) f(a) da \tag{43}$$

C Proofs and derivation

Proof of Lemma 1

Given the production technology of intermediate good production, firm ω has the following marginal cost:

$$c(\omega) = w^{\beta} \left(\int_{\omega' \in \mathcal{S}(\omega)} p(\omega', \omega)^{1-\sigma} d\omega' \right)^{\frac{1-\beta}{1-\sigma}}$$

Inserting $p(\omega', \omega) = \mu c(\omega')$ and rewriting the integral using the matched product density $m(a', a)$, we get

$$c\left(a\right) = w^{\beta} \left(\int \left(\mu c(a')\right)^{1-\sigma} m(a', a) da' \right)^{\frac{1-\beta}{1-\sigma}}$$

Using Shephard's Lemma, Hicksian demand for ω from ω' can be obtained by

$$\begin{aligned}
x(\omega, \omega') &= \frac{\partial c(\omega')x(\omega')}{\partial p\left(\omega\right)} \\
&= \frac{\partial \log c(\omega')x(\omega')}{\partial p\left(\omega\right)} c(\omega')x(\omega') \\
&= \frac{\partial}{\partial p\left(\omega\right)} \left[\log \left\{ w^{\beta} \left(\int_{\omega'' \in \mathcal{S}(\omega')} p(\omega'')^{1-\sigma} d\omega'' \right)^{\frac{1-\beta}{1-\sigma}} x(\omega') \right\} \right] c(\omega')x(\omega') \\
&= \frac{1-\beta}{1-\sigma} \frac{\partial}{\partial p\left(\omega\right)} \left[\log \left(\int_{\omega'' \in \mathcal{S}(\omega')} p(\omega'')^{1-\sigma} d\omega'' \right) \right] c(\omega')x(\omega') \\
&= \frac{1-\beta}{1-\sigma} \frac{\left(1-\sigma\right) p(\omega)^{-\sigma}}{\int_{\omega'' \in \mathcal{S}(\omega')} p(\omega'')^{1-\sigma} d\omega''} c(\omega')x(\omega') \\
&= \left(1-\beta\right) \frac{p(\omega)^{-\sigma}}{\int_{\omega'' \in \mathcal{S}(\omega')} p(\omega'')^{1-\sigma} d\omega''} c(\omega')x(\omega') \qquad (44)
\end{aligned}$$

Note that

$$\begin{aligned}
c(\omega) &= w^{\beta} \left(\int_{\omega' \in \mathcal{S}(\omega)} p(\omega')^{1-\sigma} d\omega' \right)^{\frac{1-\beta}{1-\sigma}} \\
c(\omega)^{\frac{1-\sigma}{1-\beta}} &= w^{\frac{\beta}{1-\beta}(1-\sigma)} \int_{\omega' \in \mathcal{S}(\omega)} p(\omega')^{1-\sigma} d\omega' \\
\int_{\omega' \in \mathcal{S}(\omega)} p(\omega')^{1-\sigma} d\omega' &= w^{\frac{\beta}{1-\beta}(\sigma-1)} c(\omega)^{\frac{1-\sigma}{1-\beta}}
\end{aligned}$$

Inserting this into (44),

$$x(\omega,\omega') = \left(1-\beta\right) p(\omega)^{-\sigma} c(\omega') x(\omega') w^{-\frac{\beta}{1-\beta}(\sigma-1)} c(\omega')^{\frac{\sigma-1}{1-\beta}}$$
$$x(\omega,\omega') = \left(1-\beta\right) p(\omega)^{-\sigma} w^{-\frac{\beta}{1-\beta}(\sigma-1)} c(\omega')^{\frac{\sigma-\beta}{1-\beta}} x(\omega')$$

From the good market clearing condition,

$$\begin{aligned} x(\omega) &= \int_{\omega'\in\mathcal{B}(\omega)} x(\omega,\omega') d\omega' + y(\omega) \\ &= \int_{\omega'\in\mathcal{B}(\omega)} \left(1-\beta\right) p(\omega)^{-\sigma} w^{-\frac{\beta}{1-\beta}(\sigma-1)} c(\omega')^{\frac{\sigma-\beta}{1-\beta}} x(\omega') d\omega' + y(\omega) \\ &= \left(1-\beta\right) p(\omega)^{-\sigma} w^{-\frac{\beta}{1-\beta}(\sigma-1)} \int_{\omega'\in\mathcal{B}(\omega)} c(\omega')^{\frac{\sigma-\beta}{1-\beta}} x(\omega') d\omega' + Y p(\omega)^{-\sigma} P^{\sigma} \end{aligned}$$

Using the matched product distribution function

$$x(a) = \left(1-\beta\right) p(a)^{-\sigma} w^{-\frac{\beta}{1-\beta}(\sigma-1)} \int c(a')^{\frac{\sigma-\beta}{1-\beta}} x(a') m(a',a) da' + Y p(a)^{-\sigma} P^{\sigma}$$

Denote $D(a) = x(a) P^{1-\sigma} p(a)^{\sigma}$ as demand shifter for a,

$$D(a) = \left(1-\beta\right) \mu^{-\sigma} \int \left[\frac{c(a')}{w}\right]^{\frac{\beta}{1-\beta}(\sigma-1)} D(a') m(a',a) da' + PY$$

Finally, the ideal price index P is given by

$$P = \left(\int_{\omega\in\Omega} p(\omega)^{1-\sigma} d\omega\right)^{\frac{1}{1-\sigma}}$$
$$P = \left(\int \left(\mu c(a)\right)^{1-\sigma} dF(a)\right)^{\frac{1}{1-\sigma}}$$

Labor Demand and Labor Market Clearing for Production Worker

Using Shephard's Lemma, labor demand from ω can be obtained by

$$\begin{aligned} l(\omega) &= \frac{\partial c(\omega)}{\partial w} x(\omega) \\ &= \frac{\partial \log c(\omega) x(\omega)}{\partial w} c(\omega) x(\omega) \end{aligned}$$

$$= \frac{\partial}{\partial w}\left[\log\left\{w^{\beta}\left(\int_{\omega'\in\mathcal{S}(\omega)} p(\omega')^{1-\sigma}d\omega'\right)^{\frac{1-\beta}{1-\sigma}} x(\omega)\right\}\right] c(\omega)x(\omega)$$
$$= \beta\frac{\partial}{\partial w}\left[\log w\right] c(\omega)x(\omega)$$
$$= \frac{\beta c(\omega)x(\omega)}{w}$$

With a notation,

$$\begin{aligned} wl(a) &= \beta c(a)x(a) \\ &= \beta c(a)D(a)P^{\sigma-1}p(a)^{-\sigma} \\ &= \beta\mu^{-\sigma}\left(\frac{c(a)}{P}\right)^{1-\sigma} D(a) \end{aligned}$$

From unskilled labor market clearing condition,

$$w = \beta\mu^{-\sigma}\int\left(\frac{c(a)}{P}\right)^{1-\sigma} D(a)dF(a)$$

Decomposition of Value Function

Conjecture that the value function takes an additive form

$$V^{F}(n,a) = nV(a).$$

Then, 10 can be expressed by

$$rnV(a) - nV_t(a) = n\pi(a) - \delta_F nV(a) + nV_a(a) + \max_{\lambda\geq 0}\left[n\lambda V(a) - nw_H\phi(\lambda)\right]$$

Dividing both side by n,

$$r(t)V(a) - V_t(a) = \pi(a) - \delta_F V(a) + V_a(a) + \max_{\lambda\geq 0}\left[\lambda V(a,t) - w_H\phi(\lambda)\right]$$

Proof of Lemma 2.

Proof. Let define $\mu_p(a) = Y^{\frac{1}{\sigma}}c^{SP}(a)$ and $\mu_l = Y^{\frac{1}{\sigma}}$ as Lagrangian multipliers for (19) and (21). Then, first order conditions are

$$y(a) = c^{SP}(a)^{-\sigma}, \tag{45}$$

$$l(a) = \beta c(a)^{SP}x(a), \tag{46}$$

and

$$x(a', a) = \left(1-\beta\right)\left[c(a)^{SP}\right]^{-\sigma}\left[c(a')^{SP}\right]^{\frac{\sigma-\beta}{1-\beta}} x(a') \tag{47}$$

By substituting (46) and (47) into (19), we have

$$x(a) = \frac{1}{\beta^{\beta}(1-\beta)^{1-\beta}}\left[\beta c^{SP}(a)x(a)\right]^{\beta}\left(\int\left[(1-\beta)c^{SP}(a)^{-\sigma}c^{SP}(a')^{\frac{\sigma-\beta}{1-\beta}}x(a')\right]^{\frac{\sigma-1}{\sigma}} m(a', a)da'\right)^{\frac{\sigma}{\sigma-1}(1-\beta)}.$$

Cancelling $x(a)$ from both sides and rearranging yields

$$c^{SP}(a) = \left(\int\left[c^{SP}(a')\right]^{1-\sigma} m(a', a)da'\right)^{\frac{1-\beta}{1-\sigma}},$$

Next, by substituting (46) and (47) into (20), and with similar step in the proof of Lemma 1, we could derive:

$$D^{SP}(a) = \left(1-\beta\right)\int\left[c^{SP}(a')\right]^{\frac{\beta}{1-\beta}(\sigma-1)} D^{SP}(a')m(a', a)da' + 1,$$

where $D^{SP}(a) = x(a)\left[P^{SP}\right]^{1-\sigma}\left[c^{SP}(a)\right]^{\sigma}$. Finally, from (21) we have:

$$1 = \beta\int\left(\frac{c(a)^{SP}}{P^{SP}}\right)^{1-\sigma} D^{SP}(a)f(a)da$$

□

D Social Planner's Problem

Objective function

From Lemma 2, using 24

$$\begin{aligned}
&\text{argmax}\int_0^{\infty}\exp\left(-\rho t\right)\log\left(\int y(a)^{\frac{\sigma-1}{\sigma}}dF(a)\right)^{\frac{\sigma-1}{\sigma}} dt\\
&\text{argmax}\int_0^{\infty}\exp\left(-\rho t\right)\frac{\sigma-1}{\sigma}\log\underbrace{\left(\int c^{SP}(a)^{1-\sigma}dF(a)\right)}_{P^{1-\sigma}} dt\\
=&\text{argmax}\int_0^{\infty}\exp\left(-\rho t\right)\log\left(\beta\int c^{SP}(a)^{1-\sigma}D^{SP}(a)f(a)da\right) dt
\end{aligned}$$

$$=\text{argmax} \int_0^\infty \exp(-\rho t) \log\left(\int c^{SP}(a)^{1-\sigma} D^{SP}(a) f(a) da\right) dt$$

Let

$$\mathbf{x}(t) = \left\{ f(a,t), m(a',a,t), c^{SP}(a',t)^{1-\sigma}, D(a,t), \lambda(a,t) \right\}_{a,a'\geq 0}$$
$$\boldsymbol{\mu}(t) = \left\{ \mu_f(a,t), \mu_m(a',a,t), \mu_{mf}(a',t), \mu_c(a,t), \mu_D(a,t), \mu_\lambda \right\}_{a,a'\geq 0}$$

where $\mathbf{x}(t)$ is a set of control variables and $\boldsymbol{\mu}(t)$ is a set of shadow values.

The Hamiltonian is

$$\begin{aligned}
&\mathcal{H}(t, \mathbf{x}(t), \boldsymbol{\mu}(t)) = \\
&\log\left(\int c^{SP}(a,t)^{1-\sigma} D^{SP}(a,t) f(a,t) da\right) \\
&+ \int \mu_f(a,t) \left[-\frac{\partial}{\partial a} f(a,t) + (\lambda(a,t) - \delta_F) f(a,t) + \lambda_E \boldsymbol{\delta}(a)\right] da \\
&+ \int\int \mu_m(a',a,t) \left[-\frac{\partial}{\partial a} m(a',a,t) - \frac{\partial}{\partial a'} m(a',a,t) + (\lambda(a',t) - \delta_F - \delta_M) m(a',a,t) + \zeta \frac{f(a',t)}{N_f} + \frac{\zeta_0 \lambda_E}{N_f} \boldsymbol{\delta}(a')\right] da' da \\
&+ \int \mu_{mf}(a,t) \left[\zeta_0 \frac{f(a,t)}{N_f} - m(a,0,t)\right] da \\
&+ \int \mu_D(a,t) \left[(1-\beta) \int \left[c^{SP}(a',t)\right]^{\frac{\beta}{1-\beta}(\sigma-1)} D^{SP}(a') m(a',a,t) da' - D^{SP}(a,t)\right] da \\
&+ \int \mu_c(a,t) \left[\left(\int c^{SP}(a',t)^{1-\sigma} m(a',a,t) da'\right)^{1-\beta} - c^{SP}(a,t)^{1-\sigma}\right] da \\
&+ \mu_\lambda(t) \left[1 - \int \phi(\lambda(a,t)) f(a,t) da\right]
\end{aligned}$$

In the following, we focus on the stationary version of the social planner's problem. Note that

$$\begin{aligned}
\frac{c^{SP}(a)^{1-\sigma} D^{SP}(a)}{\int c^{SP}(a)^{1-\sigma} D^{SP}(a) f(a) da} &= \beta \frac{c^{SP}(a)^{1-\sigma} D^{SP}(a)}{(P^{SP})^{1-\sigma}} \\
&= \beta \left(\frac{c^{SP}(a)}{P^{SP}}\right)^{1-\sigma} D^{SP}(a)
\end{aligned}$$

Next, we conjecture

$$\mu_D(a) = \left(\frac{c^{SP}(a)}{P^{SP}}\right)^{1-\sigma} f(a)$$

To verify that, taking a first order condition w.r.t. $c(a)^{1-\sigma}$,

$$0 = \frac{D^{SP}(a)f(a)}{\int c^{SP}(a)^{1-\sigma}D^{SP}(a)f(a)da} - \beta\int \mu_D\left(a'\right)\left[c^{SP}(a)\right]^{-\frac{\sigma-1}{1-\beta}} D^{SP}(a)m(a',a)da'$$
$$= \beta\frac{D^{SP}(a)f(a)}{\left(P^{SP}\right)^{1-\sigma}} - \beta\int \frac{c^{SP}(a')^{1-\sigma}f(a')}{\left(P^{SP}\right)^{1-\sigma}}\left[c^{SP}(a)\right]^{-\frac{\sigma-1}{1-\beta}} D^{SP}(a)m(a',a)da'$$

So

$$\beta D^{SP}(a) = \beta\int c^{SP}(a')^{1-\sigma}\left[c^{SP}(a)\right]^{-\frac{\sigma-1}{1-\beta}} D^{SP}(a)\frac{m(a',a)f(a')}{f(a)}da'$$
$$= \beta\left[c^{SP}(a)\right]^{-\frac{\sigma-1}{1-\beta}} D^{SP}(a)\int c^{SP}(a')^{1-\sigma}m(a',a)da'$$
$$= \beta D^{SP}(a)$$

which is a desired result.

First order conditions w.r.t. $D^{SP}(a)$ gives

$$\mu_D\left(a\right) = \frac{c^{SP}(a)^{1-\sigma}f(a)}{\int c^{SP}(a)^{1-\sigma}D^{SP}(a)f(a)da} + \left(1-\beta\right)\int \mu_D\left(a'\right)\left[c^{SP}(a)\right]^{\frac{\beta(\sigma-1)}{1-\beta}} m(a',a)da'.$$

So,

$$\left(\frac{c^{SP}(a)}{P^{SP}}\right)^{1-\sigma} = \beta\left(\frac{c^{SP}(a)}{P^{SP}}\right)^{1-\sigma} + \left(1-\beta\right)\int\left(\frac{c^{SP}(a')}{P^{SP}}\right)^{1-\sigma}\left[c^{SP}(a)\right]^{\frac{\beta(\sigma-1)}{1-\beta}}\frac{f(a')m(a',a)}{f(a)}da'.$$

So,

$$c^{SP}(a)^{\frac{1-\sigma}{1-\beta}} = \int c^{SP}(a')^{1-\sigma}\frac{f(a')m(a',a)}{f(a)}da'$$

So, we recovered cost functions:

$$c^{SP}(a)^{1-\sigma} = \left(\int c^{SP}(a')^{1-\sigma}m(a',a)da'\right)^{1-\beta} \tag{48}$$

First order conditions w.r.t. $f(a)$ gives

$$\rho\mu_f\left(a\right) =$$
$$\beta\left(\frac{c^{SP}(a)}{P^{SP}}\right)^{1-\sigma} D^{SP}(a) + \frac{\partial}{\partial a}\mu_f\left(a\right) + \mu_f\left(a\right)\left(\lambda(a)-\delta_F\right) - \mu_\lambda\phi\left(\lambda(a)\right) + \frac{\zeta}{N_f}\int \mu_m\left(a,a'\right)da' \tag{49}$$

First order conditions w.r.t. $m(a', a)$ gives

$$\begin{aligned}\rho\mu_m\left(a', a\right) =& \frac{\partial}{\partial a}\mu_m\left(a', a\right) + \frac{\partial}{\partial a'}\mu_m\left(a', a\right) + \mu_m\left(a', a\right)\left(\lambda\left(a'\right) - \delta_F - \delta_M\right) \\ &+ \mu_D\left(a\right)\left(1-\beta\right)\left[c^{SP}(a')\right]^{\frac{\beta(\sigma-1)}{1-\beta}} D^{SP}(a')\end{aligned}$$

So,

$$\begin{aligned}\rho\mu_m\left(a', a\right) =& \frac{\partial}{\partial a}\mu_m\left(a', a\right) + \frac{\partial}{\partial a'}\mu_m\left(a', a\right) + \mu_m\left(a', a\right)\left(\lambda\left(a'\right) - \delta_F - \delta_M\right) \\ &+ \left(1-\beta\right)\left[c^{SP}(a')\right]^{\frac{\beta(\sigma-1)}{1-\beta}} D^{SP}(a')\left(\frac{c^{SP}(a)}{P^{SP}}\right)^{1-\sigma} f(a)\end{aligned}$$

So,

$$\begin{aligned}\left(\rho + \delta_F + \delta_M - \lambda\left(a'\right)\right)\mu_m\left(a', a\right) =& \frac{\partial}{\partial a}\mu_m\left(a', a\right) + \frac{\partial}{\partial a'}\mu_m\left(a', a\right) \\ &+ c^{SP}(a)^{1-\sigma}\left(1-\beta\right)\left[c^{SP}(a')\right]^{\frac{\sigma-1}{1-\beta}} f(a)\underbrace{\left(\frac{c^{SP}(a')}{P^{SP}}\right)^{1-\sigma} D^{SP}(a')}_{\text{revenue of a'}}\end{aligned} \tag{50}$$

First order conditions w.r.t. $\lambda(a)$ gives

$$0 = \mu_f\left(a\right) f\left(a\right) + \int \mu_m\left(a, a'\right) m(a, a') da' - \mu_\lambda \phi'\left(\lambda(a)\right) f(a)$$

So,

$$\lambda(a) = \phi'^{-1}\left(\frac{1}{\mu_\lambda}\left\{\mu_f\left(a\right) + \frac{\int \mu_m\left(a, a'\right) m(a, a') da'}{f\left(a\right)}\right\}\right)$$

Note that

$$\phi'^{-1}(x) = \left(\frac{\phi}{\gamma}x\right)^{\frac{1}{\gamma-1}}$$

Therefore,

$$\lambda(a) = \left[\frac{\phi}{\gamma \mu_\lambda} \left\{ \mu_f(a) + \frac{\int \mu_m(a,a') m(a,a') da'}{f(a)} \right\} \right]^{\frac{1}{\gamma-1}}$$

Define

$$\begin{aligned} V^{SP}(a) &\equiv \mu_f(a) + \frac{\int \mu_m(a,a') m(a,a') da'}{f(a)} \\ V^M(a',a) &\equiv \mu_m(a',a) \\ w_H^{SP} &\equiv \mu_\lambda \end{aligned}$$

From (50), (changing a and a')

$$\begin{aligned} \left(\rho + \delta_F + \delta_M - \lambda(a)\right) \mu_m(a,a') =& \frac{\partial}{\partial a} \mu_m(a,a') + \frac{\partial}{\partial a'} \mu_m(a,a') \\ &+ c^{SP}(a')^{1-\sigma} \left(1-\beta\right) \left[c^{SP}(a)\right]^{\frac{\sigma-1}{1-\beta}} f(a') \left(\frac{c^{SP}(a)}{P^{SP}}\right)^{1-\sigma} D^{SP}(a) \end{aligned}$$

Multiply $m(a,a')$ and divide by $f(a)$ and integrate over a'

$$\begin{aligned} &\left(\rho + \delta_F + \delta_M - \lambda(a)\right) \int \mu_m(a,a') \frac{m(a,a')}{f(a)} da' = \\ &\frac{\partial}{\partial a} \int \mu_m(a,a') \frac{m(a,a')}{f(a)} da' \\ &+ \left(1-\beta\right) r^{SP}(a) \int \frac{c^{SP}(a')^{1-\sigma}}{\int c^{SP}(a'')^{1-\sigma} m(a'',a) da'} \frac{m(a,a') f(a')}{f(a)} da' \end{aligned}$$

So

$$\begin{aligned} \left(\rho + \delta_F + \delta_M - \lambda(a)\right) \int \mu_m(a,a') \frac{m(a,a')}{f(a)} da' =& \frac{\partial}{\partial a} \int \mu_m(a,a') \frac{m(a,a')}{f(a)} da' \\ &+ \left(1-\beta\right) r^{SP}(a) \underbrace{\frac{\int c^{SP}(a')^{1-\sigma} m(a',a) da'}{\int c^{SP}(a'')^{1-\sigma} m(a'',a) da'}}_{=1} \end{aligned}$$

So,

$$\left(\rho+\delta_F+\delta_M-\lambda\left(a\right)\right)\int\mu_m\left(a,a'\right)\frac{m(a,a')}{f(a)}da'=\frac{\partial}{\partial a}\int\mu_m\left(a,a'\right)\frac{m(a,a')}{f(a)}da' \\ +\left(1-\beta\right)r^{SP}(a), \tag{51}$$

From (49) + (51)

$$\left(\rho+\delta_F-\lambda(a)\right)V^{SP}(a)=r^{SP}(a)+\frac{\partial}{\partial a}V^{SP}(a)-w_H^{SP}\phi\left(\lambda(a)\right)+\int V^M\left(a,a'\right)\left\{\frac{\zeta}{N_f}-\delta_M\frac{m(a,a')}{f(a)}\right\}da',$$

where

$$r^{SP}(a)=\left(\frac{c^{SP}(a)}{P^{SP}}\right)^{1-\sigma}D^{SP}(a).$$

From (50),

$$\left(\rho+\delta_F+\delta_M-\lambda\left(a\right)\right)V^M\left(a,a'\right)=\frac{\partial}{\partial a}V^M\left(a,a'\right)+\frac{\partial}{\partial a'}V^M\left(a,a'\right) \\ +c^{SP}(a')^{1-\sigma}\left(1-\beta\right)\left[c^{SP}(a)\right]^{\frac{\sigma-1}{1-\beta}}f(a')r^{SP}(a)$$

$$\left(\rho+\delta_F+\delta_M-\lambda\left(a'\right)\right)\mu_m\left(a',a\right)=\frac{\partial}{\partial a}\mu_m\left(a',a\right)+\frac{\partial}{\partial a'}\mu_m\left(a',a\right) \\ +c^{SP}(a)^{1-\sigma}\left(1-\beta\right)\left[c^{SP}(a')\right]^{\frac{\sigma-1}{1-\beta}}f(a)\underbrace{\left(\frac{c^{SP}(a')}{P^{SP}}\right)^{1-\sigma}D^{SP}(a')}_{\text{revenue of a'}} \tag{52}$$

Inserting (48),

$$\left(\rho+\delta_F+\delta_M-\lambda\left(a\right)\right)V^M\left(a,a'\right)=\frac{\partial}{\partial a}V^M\left(a,a'\right)+\frac{\partial}{\partial a'}V^M\left(a,a'\right)+r^{SP}(a)\left(1-\beta\right)\Omega(a',a)f(a')$$

where

$$\Omega(a',a)=\frac{c^{SP}(a')^{1-\sigma}}{\int c^{SP}(a'')^{1-\sigma}m(a'',a)da''}$$

E Numerical Appendix

Product Distribution f

We solve the model with finite difference methods. Throughout this section, to construct the derivative matrices, we use a backward approximation when the drift of the state variable is positive, and a forward approximation when the drift of the state is negative. Notice that the stationary distribution is the solution for the following differential equation:

$$0 = -\frac{\partial f\left(a\right)}{\partial a} + \left(\lambda(a) - \delta_F\right) f\left(a\right) + \lambda_E \boldsymbol{\delta}(a) \tag{53}$$

We now discretize a on an evenly spaced $N_a \times 1$. Let D_a be the $N_a \times N_a$ matrix that, when pre-multiplying f, gives an approximation of f_a. Analogously, define D_a:

$$f_a = D_a f$$

Vectorize (53) and obtain,

$$f = -\left\{-D_a + \lambda - \delta_F\right\}^{-1} \lambda_E \boldsymbol{\delta}$$

where the element of N_a vector f consists of $f\left(a\right)$, the element of N_a vector λ consists of $\lambda\left(a\right)$, and the element of N_a vector $\boldsymbol{\delta}$ consists of $\boldsymbol{\delta}(a)$

Matched Product Distribution m

The distributions of matched products $m(a', a)$ is given by

$$\frac{\partial}{\partial a} m(a', a) = -\frac{\partial}{\partial a'} m(a', a) + \left(\lambda\left(a'\right) - \delta_F - \delta_M\right) m(a', a) + \frac{\zeta_0 \lambda_E}{N} f(a') + \zeta_0 \frac{\lambda_E}{N_f} \boldsymbol{\delta}(a')$$

subject to the boundary condition $m(a'; 0) = 0$. Vectorize this, and obtain

$$\begin{aligned}
\frac{m - m_{-da}}{da} &= \left(-D_a + \lambda\left(a'\right) - \delta_F - \delta_M\right) m + \frac{\zeta_0 \lambda_E}{N} f + \zeta_0 \frac{\lambda_E}{N_f} \boldsymbol{\delta} \\
m - m_{-da} &= da\left(-D_a + \lambda\left(a'\right) - \delta_F - \delta_M\right) m + da \frac{\zeta_0 \lambda_E}{N} f + da \zeta_0 \frac{\lambda_E}{N_f} \boldsymbol{\delta} \\
m &= \left\{I_a - da\left(-D_a + \lambda\left(a'\right) - \delta_F - \delta_M\right)\right\}^{-1} \left(m_{-da} + da \frac{\zeta_0 \lambda_E}{N} f + da \zeta_0 \frac{\lambda_E}{N_f} \boldsymbol{\delta}\right)
\end{aligned}$$

Starting from $m(\cdot;0) = 0$, the forward iteration of the above vectorized equation gives the distributions of matched products $m(a', a)$ for each a. Note that $m(a', a)$ converges as a becomes sufficiently large. Therefore, we only need to forward iterate until $m(a', a)$ converges.

Value Function V

Let Δ denote step-size and τ the iteration of the algorithm. Then given $V^{\tau-1}(a)$, the implicit method gives an update

$$\frac{1}{\Delta}\left(V^{\tau}(a) - V^{\tau-1}(a)\right) + \left(\rho + \delta_F\right)V^{\tau}(a) = \pi(a) + V_a^{\tau}(a) + \lambda(a)V^{\tau}(a) - w_H\phi(\lambda(a))$$

where $\pi(a) = \left(1 - \frac{1}{\mu}\right)\left(\frac{\mu c(a)}{P}\right)^{1-\sigma} D(a)$. Rearranging this,

$$\left(\frac{1}{\Delta} + \rho + \delta_F - \lambda(a)\right)V^{\tau}(a) - V_a^{\tau}(a) = \pi(a) - w_H\phi(\lambda(a)) + \frac{1}{\Delta}V^{\tau-1}(a)$$

Now, we vectorize the HJB equation:

$$\left(\frac{1}{\Delta} + \rho + \delta_F - \Lambda - D_a\right)V^{\tau} = \pi - w_H\phi(\lambda) + \frac{1}{\Delta}V^{\tau-1}$$

$$V^{\tau} = \left\{\frac{1}{\Delta} + \rho + \delta_F - \Lambda - D_a\right\}^{-1}\left(\pi - w_H\phi(\lambda) + \frac{1}{\Delta}V^{\tau-1}\right)$$

where the element of the N_a-dimensional vector V^{τ} consists of $V^{\tau}(a)$, the element of the N_a-dimensional vector π consists of $\pi(a)$, $\Lambda \equiv \text{diag}(\lambda(a_i))$ collects the age-specific innovation rates along the grid $\{a_i\}_{i=1}^{N_a}$, and $\phi(\lambda)$ denotes the vector whose entries are $\phi(\lambda(a_i))$. The implicit method works by updating V^{τ} through the above equation.

Social Matching Value Function V^M

$$\left(\rho + \delta_F + \delta_M - \lambda(a)\right)V^M(a, a') = \frac{\partial}{\partial a}V^M(a, a') + \frac{\partial}{\partial a'}V^M(a, a') + \left(1 - \beta\right)r^{SP}(a)\Omega(a', a)f(a')$$

We now discretize a and a' on evenly spaced grids with N_a nodes each. Stack these according to:

$$\begin{pmatrix} a_1, a'_1 \\ a_2, a'_1 \\ \vdots \\ a_{N_a}, a'_1 \\ \vdots \\ a_1, a'_{N_a} \\ \vdots \\ a_{N_a}, a'_{N_a} \end{pmatrix}$$

$$\begin{aligned} \left(\rho + \delta_F + \delta_M - \Lambda_a - D_a - D_{a'}\right) V^M &= A \\ V^M &= \left(\rho + \delta_F + \delta_M - \Lambda_a - D_a - D_{a'}\right)^{-1} A \end{aligned}$$

where the N_a^2-dimensional vector V^M collects $V^M(a, a')$ evaluated on the stacked grid, $\Lambda_a \equiv I_{N_a} \otimes \Lambda$ applies the age-specific innovation rates to the first argument, D_a and $D_{a'}$ are the finite-difference operators with respect to a and a', and the N_a^2-dimensional vector A has elements

$$A = \left(1 - \beta\right) r^{SP}(a)\Omega(a', a) f(a') \quad \text{for each stacked pair } (a, a').$$